University Mathematics

A TEXTBOOK FOR STUDENTS OF SCIENCE AND ENGINEERING

BY

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PREFACE

This book is intended as a first-year course in Pure Mathematics at any University and contains practically all the various branches of Mathematics required by students, excluding projective geometry, although the analysis is not intended to be rigorous. It would also be very suitable for advanced Sixth Form students in Grammar Schools, especially those entering for special scholarship examinations.

As practically the whole of the problems are taken from London University Science and Engineering Degree examination papers in Pure Mathematics, and since the book covers the whole of the syllabus for the London General Degree in Science (which covers Subsidiary Mathematics), excluding projective geometry which is optional, it will be found eminently suitable for this course.

My grateful thanks are due to the Senate of the London University for permission to use examples from their final degree examination papers, and also to an old student, Mr. D. Stewart, for his assistance in the preparation of the manuscript.

Even after the most careful checking it is quite possible that some errors have been overlooked, and I shall be greatly indebted to anyone submitting corrections.

JOSEPH BLAKEY.

CONSTANTINE TECHNICAL COLLEGE,
MIDDLESBROUGH,
1949.

CONTENTS

2. (i) Find the ranges of values of x for which the expression $x^2 - 6x + 7$ lies in value between ± 1.

(ii) Find the least value of k so that the expression

$$3x^2 + 12xy + 7y^2 + k(x^2 + y^2)$$

shall be greater than zero for all real values of x and y.

3. Show that, if a and b have opposite signs, the expression $ax + b/x$ can assume all real values; but that if a and b are of the same sign, it cannot assume any value lying between $\pm 2\sqrt{(ab)}$.

Express $\dfrac{x^2 + 1}{x(x - 1)}$ as a function of y, where $x - 1 = xy$, and hence, or otherwise, show that the expression can only assume values which do not lie between $-2 \pm 2\sqrt{2}$.

4. Find the condition that the expression $ax^2 + bx + c$ should have the same sign for all values of x.

Show that, for real values of x, the expression $(ax^2 + bx + c)/(cx^2 + bx + a)$ will be capable of all real values if $b^2 > (a + c)^2$.

Show also that there will be two values between which it cannot lie if $4ac < b^2 < (a + c)^2$, and two values between which it must lie if $b^2 < 4ac$.

5. Show that, if $y = (2x^2 - 4)/(x^2 - x - 2)$, y can assume all real values for real values of x, and find the ranges of values of x for which $y > 1$.

Draw a rough graph of the function $(2x^2 - 4)/(x^2 - x - 2)$.

6. Find the conditions that the expression $ax^2 + bx + c$ should be positive for all real values of x.

Show that the expression $(x^2 - 3ax + 2a^2)/(x^2 - 3x + 2)$, where $a \neq 1$, can assume any real value for real values of x only if $\frac{1}{2} \leqslant a \leqslant 2$.

Show that, if $a = 0$, there will be two extreme values between which the expression cannot lie, and determine these values.

7. Prove that, if $a + b + c = 0$ and $bc + ca + ab + 3m = 0$, then the expression E, where $E = (x^2 + ax + m)(x^2 + bx + m)(x^2 + cx + m)$, will contain no powers of x except those whose index is a multiple of three.

Given that the expression $x^6 + 16x^3 + 64$ has a factor of the form $x^2 - 2x + m$, resolve it into three quadratic factors similar to E, and deduce all the roots of the equation $x^6 + 16x^3 + 64 = 0$.

8. (i) If α, β are the roots of the equation $ax^2 + 2x + b = 0$, evaluate $(a^2\alpha^4 - b^2)(a^2\beta^4 - b^2)$.

(ii) Prove that, whatever real values x may take, the value of $\dfrac{5x^2 + 8x + 5}{4x^2 + 10x + 4}$ cannot lie between ± 1.

9. (i) Show that, if $\dfrac{ax^2 + 2bx + c}{cx^2 + 2bx + a}$, where a, b, c are positive, can assume all possible real values, then $(a + c)^2 < 4b^2$.

(ii) Draw a rough graph of the function $\dfrac{4x^2 + 6x + 1}{x^2 + 3x + 2}$ showing clearly the form of the graph when x is numerically large.

Find the range of values of x for which the value of the function is greater than 4.

10. Sum to n terms the series whose rth term is $(2r + 1)3^r$. [*Hint.*—Series is an arithmetico-geometric one, and the method of finding the sum is the same as for developing the formula for a G.P.]

11. Determine the coefficients A, B, C, so that, if $f(x)$ denotes the polynomial $Ax^5 + Bx^3 + Cx$, then $f(x) - f(x - 1) = (2x - 1)^4$, for all values of x.

Find the sum of the fourth powers of the first n *odd* integers (positive).

12. If $(1 + x)^n = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, where n is a positive integer, find the value of $a_0^2 + a_1^2 + \dots + a_n^2$, and prove that

$$a_1^2 + 2a_2^2 + 3a_3^2 + \dots + na_n^2 = \frac{n}{2}(a_0^2 + a_1^2 + \dots + a_n^2).$$

Show also that $a_1 + 2a_2 + 3a_3 + \dots + na_n = n \,.\, 2^{n-1}$.

13. If $A + B + C = \pi$, prove that

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1.$$

A hexagon, two of whose sides are of length $2x$, two of length $2y$, and two of length $2z$, is inscribed in a circle. Prove that the radius r of this circle is given by the equation $r^3 - (x^2 + y^2 + z^2)r - 2xyz = 0$.

14. A, B, C are three points in order on a straight line so that AB $= 2a$, BC $= 2b$. Semicircles are drawn on AB, BC, CA as diameters, on the same side of the line ABC. Show that the radius of the circle drawn to touch all three semicircles is $ab(a + b)/(a^2 + ab + b^2)$.

15. A triangle, two of whose sides are a and b ($a > b$), is inscribed in a circle of diameter d. Show that

(i) $2d^2\Delta = ab(a\sqrt{d^2 - b^2} \pm b\sqrt{d^2 - a^2})$.

(ii) $\angle\text{BCA} = \cos^{-1}\left\{\dfrac{ab \pm (\sqrt{d^2 - a^2)(d^2 - b^2})}{d^2}\right\}$.

16. If AD, BE, CF are the altitudes of a given acute-angled triangle ABC find the angles and lengths of the triangle DEF.

Prove that, with the usual standard notation,

(i) area of $\triangle$DEF $= 2\Delta \cos A \cos B \cos C$.

(ii) perimeter of $\triangle$DEF $= abc/2R^2$.

17. I_1, I_2, I_3 are the centres of the escribed circles of the triangle ABC, opposite to A, B, C respectively. Show that

(i) The centre of the inscribed circle of the triangle ABC is the orthocentre of the triangle $\text{I}_1\text{I}_2\text{I}_3$.

(ii) $\text{I}_1\text{I}_2 = 4R\cos\frac{1}{2}C$, where R is the radius of the circle circumscribing triangle ABC.

(iii) The ratio of the area of the triangle ABC to the area of the triangle $\text{I}_1\text{I}_2\text{I}_3$ is $2\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$ to 1.

18. (i) If the median through the vertex A of the triangle ABC makes angles θ, ϕ respectively with the sides AB and AC, prove that

$$c\sin\theta = b\sin\phi,\text{ and }\tan\tfrac{1}{2}(\phi - \theta) = \frac{c - b}{c + b}\tan\tfrac{1}{2}A.$$

(ii) The inscribed circle of a triangle ABC touches the sides BC, CA, AB in X, Y, Z respectively. I is the centre of the circle. If XI produced meets ZY in L, prove that AL is a median of the triangle.

19. A and B are two fixed marks on one bank of a river at a known distance a apart; C, D are two points on the opposite bank also at a distance a apart, such that CD is parallel to AB and A, B, C, D are in the same horizontal plane. If AB subtends angles α and β at C and D respectively, show that the width of the river is $2a(\cot\alpha + \cot\beta)/\{4 + (\cot\alpha - \cot\beta)^2\}$.

20. Find the general value of θ satisfying the equation $2\cos 3\theta = 1$, and hence find the roots of the equation $8x^3 - 6x - 1 = 0$.

Show that (i) $\sec\frac{\pi}{9} + \sec\frac{5\pi}{9} + \sec\frac{7\pi}{9} = -6$;

(ii) $\sec^2\frac{\pi}{9} + \sec^2\frac{5\pi}{9} + \sec^2\frac{7\pi}{9} = 36$.

21. R is the circumradius of the triangle ABC and r_1, r_2, r_3 are the radii of the escribed circles.

If the distances between the centres of the escribed circles are α, β, γ, prove that

(i) $\Delta ABC = r_1r_2r_3/\sqrt{r_2r_3 + r_3r_1 + r_1r_2}$.

(ii) $8R = \dfrac{\alpha\beta\gamma}{\sqrt{\sigma(\sigma-\alpha)(\sigma-\beta)(\sigma-\gamma)}}$, where $\sigma = \frac{1}{2}(\alpha+\beta+\gamma)$.

22. ABC is an acute-angled triangle; the perpendiculars to BC, CA, AB through A, B, C, respectively, meet the sides at D, E, F, and they are concurrent at H.

Prove that, if R be the radius of the circumscribing circle of the triangle ABC,

(i) $\Delta BHC : \Delta CHA : \Delta AHB = \tan A : \tan B : \tan C$;

(ii) the radius of the circumcircle of ΔDEF is $\frac{1}{2}R$;

(iii) the radius of the inscribed circle of triangle DEF is $2R\cos A\cos B\cos C$.

23. If O, H are the circumcentre and orthocentre respectively of the triangle ABC, prove that $OH^2 = R^2(1 - 8\cos A\cos B\cos C)$, where R is the radius of the circumscribing circle.

ABC is an acute-angled triangle and X, Y, Z are the mid-points of the minor arcs BC, CA, AB of the circumscribing circle. Find the angles of the triangle XYZ and prove that I, the centre of the inscribed circle of triangle ABC, is the orthocentre of the triangle XYZ.

Hence deduce that $OI^2 = R^2(1 - 8\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C)$.

24. If PN, the perpendicular from $P \equiv (p, q)$ on to the line $lx + my + n = 0$, is produced to P′ so that NP′ = PN, find the co-ordinates of P′, the reflection of the point P in the line.

Find the equation of the reflection of the line $x + y = 1$ in the line $x + 2y = 3$.

25. ABCD is a cyclic quadrilateral. The equations of the sides AB, BC, DA are $x - 3y = 0$, $4x - 3y - 9 = 0$, $x + 4y + 4 = 0$ respectively. If the side BC is of length 5/3 units, find the equations of the two lines along which the side CD can lie.

26. Find the equation of the circle of which the points (x_1, y_1), (x_2, y_2) are the extremities of a diameter.

Find the co-ordinates of the extremities of the diameter perpendicular to the one above.

27. Write down the equation of the perpendicular bisector of the line joining (x_1, y_1), (x_2, y_2).

The equations of the perpendicular bisectors of the sides AB, AC of a triangle ABC are $x + y = 0$ and $x - 2y = 0$, and the side BC passes through the point $(1, 2)$. Show that the locus of A is the circle $x^2 + y^2 - x + 7y = 0$.

28. Show that, for all values of the constants p and q, the circle whose equation is $(x - a)(x - a + p) + (y - b)(y - b + q) = r^2$ bisects the circumference of the circle whose equation is $(x - a)^2 + (y - b)^2 = r^2$.

Find the equation of the circle that bisects the circumference of the circle $x^2 + y^2 + 2y = 3$ and touches the line $(x - y) = 0$ at the origin.

29. The line $lx + my = 0$ bisects at right angles the line joining the points $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$.

Show that $$\frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{-2(lx_1 + my_1)}{l^2 + m^2}.$$

Show that the locus of a point, which is such that its reflections in two straight lines $lx + my = 0$ and $lx - my = 0$ are collinear with a fixed point (h, k), is the circle whose equation is $(l^2 - m^2)(x^2 + y^2) + (l^2 + m^2)(hx - ky) = 0$.

CHAPTER II

Limits; Convergency and Divergency of Series; Exponential and Hyperbolic Functions; Complex Numbers

Limits

1. Consider the function $(x-1)/(x+2)$ and replace x by $3+h$ in this function. The value of the result is $(2+h)/(5+h)$ and, as h gets smaller and smaller, the value of this result gets closer and closer to 2/5. It is said that "the limiting value of $(2+h)/(5+h)$ as h approaches zero $(h \to 0)$" is 2/5, and this is stated more briefly as $\underset{h\to 0}{\mathrm{Lt}} \dfrac{2+h}{5+h} = \dfrac{2}{5}$, where "Lt" represents "the limiting value of". But as $h \to 0$ the value of $x \to 3$, and therefore $\underset{x\to 3}{\mathrm{Lt}} \dfrac{x-1}{x+2} = \dfrac{2}{5}$.

If x be given the value 3 in $(x-1)/(x+2)$ the result is also 2/5, and therefore in this case $\underset{x\to 3}{\mathrm{Lt}} \dfrac{x-1}{x+2}$ is the value of $(x-1)/(x+2)$ when $x = 3$.

Next consider the function $(x^2-9)/(x-3)$ and replace x by $3+h$ in this function. The result is $(6h+h^2)/h = 6+h$ and, as $h \to 0$, the value of this result approaches 6. Thus, as above, in this case it can be said that $\underset{x\to 3}{\mathrm{Lt}} \dfrac{x^2-9}{x-3} = 6$.

Now, when x is given the value 3 in $(x^2-9)/(x-3)$ the result is 0/0, which is indeterminate; but, if $(x^2-9)/(x-3)$ be reduced to its *lowest terms*, namely $(x+3)$, and then x be given the value 3, the result is 6, which is the same as the limiting value obtained above.

By several examples of this nature it can be shown that the limiting value of $f(x)$ where $x \to a$ can be obtained by reducing $f(x)$ to its lowest terms, and then replacing x by a in the result.

The definition that will be used for the limiting value of $f(x)$, as x approaches a, is "the finite value that the function approaches as x gets closer and closer to the value a".

If $f(x)$ approaches an infinite value as x approaches a, this will not fulfil the given definition and is not considered as a limiting value.

From all the above results it can be seen that, in order to obtain the value of $\operatorname*{Lt}_{x \to a} f(x)$, it is first necessary to reduce $f(x)$ to its lowest terms and then insert $x = a$ in the result.

For example, $\operatorname*{Lt}_{x \to 0} \dfrac{x^2 - 2x}{x} = \operatorname*{Lt}_{x \to 0} (x - 2) = 0 - 2 = -2.$

In general the functions to be dealt with will be *continuous* in the range of the variable to be considered. The function $f(x)$ is said to be continuous at $x = x_1$, if $\operatorname*{Lt}_{x \to x_1} f(x) = f(x_1)$.

When this condition does not hold, $f(x)$ is said to be *discontinuous* at $x = x_1$, and this is equivalent to saying that there will be a break in the graph of the function at $x = x_1$. Thus, $\tan x$ is discontinuous for any value of x which is an odd multiple of $\pi/2$, and $\dfrac{1}{x(x-1)}$ is discontinuous at both $x = 0$ and $x = 1$, since, when x is small and positive, $1/\{x(x-1)\}$ is a negative large number.

When x is small and negative, $1/\{x(x-1)\}$ is a positive large number.

When x is slightly greater than unity, $1/\{x(x-1)\}$ is a positive large number.

When x is slightly less than unity, $1/\{x(x-1)\}$ is a negative large number.

2. The following theorems on limits will be assumed without proof, and the variable is considered as approaching any given value:

(i) The limit of the sum of a finite number of functions is equal to the sum of their limits.

(ii) The limit of the product of a finite number of functions is equal to the product of their limits.

(iii) The limit of the quotient of two functions is equal to the quotient of their limits.

3. The following theorem on limits is very important.

Theorem.

If θ be in *radians*, then $\operatorname*{Lt}_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1$ and $\operatorname*{Lt}_{\theta \to 0} \dfrac{\tan \theta}{\theta} = 1$.

If u_n be the nth term,

$$u_n = (-1)^{n-1}\frac{x^n}{n},\quad u_{n+1} = (-1)^n\frac{x^{n+1}}{n+1},\quad \therefore\ \frac{u_{n+1}}{u_n} = \frac{-nx}{n+1}.$$

$$\left|\operatorname*{Lt}_{n\to\infty}\frac{u_{n+1}}{u_n}\right| = \left|\operatorname*{Lt}_{n\to\infty} -\frac{nx}{n+1}\right| = \operatorname*{Lt}_{n\to\infty}\left|-x\left(1-\frac{1}{n+1}\right)\right| = |x|.$$

Hence the series is *convergent if* $|x| < 1$ and *divergent if* $|x| > 1$.

If $x = 1$ the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \to \infty$, which is a series of alternately positive and negative terms descending in magnitude, and the series is thus *convergent for* $x = +1$. When x becomes -1 the series is

$$-1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{4} - \ldots > \infty = -(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots \to \infty),$$

which has been proved to be divergent, and hence the series is *divergent when* $x = -1$. Thus the logarithmic series is convergent for $-1 < x \leqslant 1$, and divergent for all other values of x.

10. Binomial Theorem for negative and fractional indices.

The truth of the binomial theorem for fractional and negative indices will be assumed, i.e. for all rational values of n it will be taken that

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \ldots$$
$$+\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}x^r + \ldots$$

The general term is taken to be the $(r+1)$th term, namely

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}x^r,$$

and it is to be noted that the degree in a and x combined is n, and there are r factors in both numerator and denominator of the coefficient.

11. Theorem.

To prove that, if n be not a positive integer, there are an infinite number of terms in the binomial expansion of $(a+x)^n$.

Consider the general term, whose coefficient is

$$n(n-1)(n-2)\ldots(n-r+1)/r!$$

This can only vanish if $n - r + 1 = 0$, i.e. if $r = n + 1$. But r is always a positive integer, so that this is only possible when n is a positive integer. Hence, there will be an infinite number of terms if n is fractional or negative.

12. Theorem.

To find for what values of x the expansion of $(a + x)^n$ is valid (i.e. convergent), given that n is fractional or negative.

If u_r be the rth term, then

$$u_r = \frac{n(n-1)(n-2)\ldots(n-r+2)}{(r-1)!}a^{n-r+1}x^{r-1},$$

$$u_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}a^{n-r}x^r.$$

$$\therefore\ \frac{u_{r+1}}{u_r} = \frac{n-r+1}{r}\cdot\frac{x}{a} = \left(\frac{n+1}{r} - 1\right)\frac{x}{a}.$$

$$\therefore\ \left|\operatorname*{Lt}_{r\to\infty}\frac{u_{r+1}}{u_r}\right| = \left|\operatorname*{Lt}_{r\to\infty}\left(\frac{n+1}{r} - 1\right)\frac{x}{a}\right| = \left|\frac{x}{a}\right|.$$

$\therefore$ the series is convergent if $|x/a| < 1$, i.e. $|x| < |a|$.

The following *standard expansions* obtained by using the binomial theorem should be memorized:

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots \to \infty. \quad \text{[G.P.]}$$

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots \to \infty. \quad \text{[G.P.]}$$

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \to \infty.$$

[Arithmetico-Geometrical.]

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots \to \infty.$$

[Arithmetico-Geometrical.]

$$1/(1-x)^3 = (1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots \to \infty.$$

$$1/(1+x)^3 = (1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots \to \infty.$$

$$n\text{th term} = \frac{n(n+1)}{2}x^{n-1}.$$

In each of these results the series is convergent if $|x| < 1$.

13. Theorem.

To prove that all the terms in the expansion of $(a - x)^{-n}$ are positive if n is positive.

The general $(r+1)$th term in the expansion of $(a-x)^{-n}$ is

$$\frac{(-n)(-n-1)(-n-2)\ldots(-n-r+1)}{r!}a^{n-r}(-x)^r$$

$$= (-1)^r n(n+1)(n+2)\ldots(n+r-1)a^{n-r}(-1)^r x^r/r!$$

$$= (-1)^{2r} n(n+1)(n+2)\ldots(n+r-1)a^{n-r}x^r/r!$$

$=$ positive quantity for all values of r, thus proving the theorem.

Note 1.—It is usually advisable to base the expansion of $(a+x)^n$ on the expansion of $(1+y)^n$, which is

$$1 + ny + \frac{n(n-1)}{2!}y^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}y^r + \ldots$$

as this result is easier to deal with.

This can always be done, since

$$(a+x)^n = \{a(1+x/a)\}^n = a^n(1+x/a)^n$$
$$= a^n\{1+y\}^n, \text{ where } y = x/a.$$

Example 4.—Expand to four terms, in ascending powers of x, the binomial expression $(2-x/2)^{\frac{1}{2}}$.

$$\left(2-\frac{x}{2}\right)^{\frac{1}{2}} = 2^{\frac{1}{2}}(1-x/4)^{\frac{1}{2}}$$

$$= 2^{\frac{1}{2}}\left[1+\tfrac{1}{2}\left(-\frac{x}{4}\right)+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}\left(-\frac{x}{4}\right)^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}\left(-\frac{x}{4}\right)^3+\ldots\right]$$

$$= 2^{\frac{1}{2}}\left[1-\frac{x}{8}-\frac{x^2}{128}-\frac{x^3}{1024}\right] \text{ to four terms.}$$

Example 5.—Expand $(8+3x)^{-\frac{1}{3}}$ to four terms in ascending powers of x.

$$(8+3x)^{-\frac{1}{3}} = 8^{-\frac{1}{3}}\left(1+\frac{3x}{8}\right)^{-\frac{1}{3}}$$

$$= \tfrac{1}{2}\left\{1+(-\tfrac{1}{3})\left(\frac{3x}{8}\right)+\frac{(-\frac{1}{3})(-\frac{4}{3})}{2!}\left(\frac{3x}{8}\right)^2+\frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{3!}\left(\frac{3x}{8}\right)^3+\ldots\right\}$$

$$= \tfrac{1}{2}\left\{1-\frac{x}{8}+\frac{2x^2}{8^2}-\frac{7x^3}{768}+\ldots\right\}$$

$$= \tfrac{1}{2}-\frac{x}{16}+\frac{x^2}{64}-\frac{7x^3}{1536} \text{ to four terms.}$$

Note 2.—When dealing with the quotients of binomial functions it is advisable to express them as products before expansion.

Example 6.—Expand in ascending powers of x as far as the term in x^3

$$\text{(i)}\ \frac{(2+x)^4}{(1-x)^3};\quad \text{(ii)}\ \frac{(1+x)^3(1+2x)^{\frac{1}{2}}}{(3+x)^2}.$$

$$\text{(i)}\ \frac{(2+x)^4}{(1-x)^3} = (2+x)^4(1-x)^{-3}$$

$$= (16 + 32x + 24x^2 + 8x^3 + \ldots) \times (1 + 3x + 6x^2 + 10x^3 + \ldots)$$

$$= (16 + 48x + 96x^2 + 160x^3 + \ldots) + (32x + 96x^2 + 192x^3 + \ldots)$$
$$+ (24x^2 + 72x^3 + \ldots) + 8x^3 + \ldots$$

$$= 16 + 80x + 216x^2 + 432x^3 \text{ as far as } x^3 \text{ term.}$$

$$\text{(ii)}\ \frac{(1+x)^3(1+2x)^{\frac{1}{2}}}{(3+x)^2}$$

$$= \frac{(1+x)^3(1+2x)^{\frac{1}{2}}}{9(1+x/3)^2} = \tfrac{1}{9}(1+x)^3(1+2x)^{\frac{1}{2}}(1+x/3)^{-2}$$

$$= \tfrac{1}{9}(1 + 3x + 3x^2 + x^3)\{1 + \tfrac{1}{2}(2x) + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}(2x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}(2x)^3 + \ldots\}$$
$$\times \{1 - 2(x/3) + 3(x/3)^2 - 4(x/3)^3 + \ldots\}$$

$$= \tfrac{1}{9}(1 + 3x + 3x^2 + x^3)\{1 + x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 + \ldots\}$$
$$\times \{1 - \tfrac{2}{3}x + \tfrac{1}{3}x^2 - \tfrac{4}{27}x^3 + \ldots\}$$

$$= \tfrac{1}{9}\{(1 + x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 + \ldots) + (3x + 3x^2 - \tfrac{3}{2}x^3 + \ldots) + (3x^2 + 3x^3 + \ldots)$$
$$+ x^3 + \ldots\} \times \{1 - \tfrac{2}{3}x + \tfrac{1}{3}x^2 - \tfrac{4}{27}x^3 + \ldots\}$$

$$= \tfrac{1}{9}\{1 + 4x + \tfrac{11}{2}x^2 + 3x^3 + \ldots\}\{1 - \tfrac{2}{3}x + \tfrac{1}{3}x^2 - \tfrac{4}{27}x^3 + \ldots\}$$

$$= \tfrac{1}{9}\{(1 - \tfrac{2}{3}x + \tfrac{1}{3}x^2 - \tfrac{4}{27}x^3 + \ldots) + (4x - \tfrac{8}{3}x^2 + \tfrac{4}{3}x^3 + \ldots)$$
$$+ (\tfrac{11}{2}x^2 - \tfrac{11}{3}x^3 + \ldots) + 3x^3 + \ldots\}$$

$$= \tfrac{1}{9}\{1 + \tfrac{10}{3}x + \tfrac{19}{6}x^2 + \tfrac{14}{27}x^3\} \text{ as far as the term in } x^3.$$

Note 3.—If x be small compared with unity, by using the binomial theorem, the following approximations can be used for $(1 + x)^n$.

First approximation $\quad 1 + nx.$

Second approximation $1 + nx + \dfrac{n(n-1)}{2!}x^2.$

Third approximation $\quad 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3.$

The higher the approximation taken the more accurate is the result, but the degree of approximation taken in any particular case is dependent upon the data given in the question, and the degree of accuracy required in the result.

Example 7.—Find to five decimal places the value of $\sqrt{3{\cdot}96}$.

$$\sqrt{3{\cdot}96} = \sqrt{4 \times 0{\cdot}99} = 2(1 - 0{\cdot}01)^{\frac{1}{2}}$$

$$= 2\left\{1 + \tfrac{1}{2}(-0{\cdot}01) + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}(-0{\cdot}01)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}(-0{\cdot}01)^3 + \ldots\right\}$$

$$= 2\{1 - 0{\cdot}005 - 0{\cdot}0000125 - 0{\cdot}0000001 + \ldots\}$$

$$= 2\{0{\cdot}9949874\} = 1{\cdot}98997 \text{ to 5 decimal places.}$$

N.B.—With the factor two outside the bracket the quantity inside the bracket must be taken to seven decimal place accuracy.

Exponentials

14. The quantity e^x, defined as being $\underset{n\to\infty}{\mathrm{Lt}}\left(1 + \frac{x}{n}\right)^n$, is known as the *exponential function*, and with this definition

$$e = \underset{n\to\infty}{\mathrm{Lt}}\left(1 + \frac{1}{n}\right)^n.$$

Since $e^x = (e)^x$, it follows that e^x is also equal to $\underset{n\to\infty}{\mathrm{Lt}}(1 + 1/n)^{nx}$ and could be defined in this manner.

Expanding by the binomial theorem

$$\left(1 + \frac{x}{n}\right)^n = 1 + n\left(\frac{x}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{x}{n}\right)^2$$

$$+ \frac{n(n-1)(n-2)}{3!}\left(\frac{x}{n}\right)^3 + \ldots \to \infty$$

$$= 1 + x + \frac{1(1-1/n)}{2!}x^2 + \frac{1(1-1/n)(1-2/n)}{3!}x^3 + \ldots \to \infty.$$

$$\therefore \underset{n\to\infty}{\mathrm{Lt}}(1 + x/n)^n$$

$$= \underset{n\to\infty}{\mathrm{Lt}}\left\{1 + x + \frac{1-1/n}{2!}x^2 + \frac{(1-1/n)(1-2/n)}{3!}x^3 + \ldots \to \infty\right\}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \to \infty.$$

$$\therefore\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \to \infty.$$

This series is known as the exponential series and was earlier proved to be convergent for all values of x.

Replacing x by $(-x)$ in this result it can also be seen that

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots \to \infty.$$

Giving x the value unity in the series for e^x

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

On calculation, from this result by arithmetical processes, it is found that the value of e is 2·71828 to five decimal places.

$$\begin{aligned}
1 + 1 &= 2{\cdot}000000 \\
\tfrac{1}{2!} &= {\cdot}500000 \\
\tfrac{1}{3!} &= {\cdot}166667 \\
\tfrac{1}{4!} = \tfrac{1}{4} \times \tfrac{1}{3!} &= {\cdot}041667 \\
\tfrac{1}{5!} = \tfrac{1}{5} \times \tfrac{1}{4!} &= {\cdot}008333 \\
\tfrac{1}{6!} = \tfrac{1}{6} \times \tfrac{1}{5!} &= {\cdot}001389 \\
\tfrac{1}{7!} = \tfrac{1}{7} \times \tfrac{1}{6!} &= {\cdot}000198 \\
\tfrac{1}{8!} = \tfrac{1}{8} \times \tfrac{1}{7!} &= {\cdot}000025 \\
\tfrac{1}{9!} = \tfrac{1}{9} \times \tfrac{1}{8!} &= {\cdot}000003 \\
\hline
&\ 2{\cdot}718282
\end{aligned}$$

15. Theorem.

To prove that

$$a^x = 1 + x\log_e a + \frac{(x\log_e a)^2}{2!} + \frac{(x\log_e a)^3}{3!} + \ldots,$$

where a is any constant.

$$\text{Let } a^x = e^y.$$

Taking logs to the base e of each side,

$$x\log_e a = y.$$

$$\therefore\ a^x = e^y = e^{x\log_e a}.$$

But
$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots \to \infty;$$

$$\therefore\quad a^x = 1 + x\log_e a + \frac{(x\log_e a)^2}{2!} + \frac{(x\log_e a)^3}{3!} + \ldots \to \infty.$$

Note.—The exponential quantity e is a highly important quantity in mathematics, and is the base of Napierian logarithms (natural logarithms).

LOGARITHMIC SERIES

16. Theorem.

To find a series for $\log_e(1 + y)$ in ascending powers of y and its range of convergency.

It has been proved that

$$a^x = 1 + x\log_e a + \frac{(x\log_e a)^2}{2!} + \frac{(x\log_e a)^3}{3!} + \ldots \to \infty.$$

Replacing a by $(1 + y)$ in this series,

$$(1 + y)^x = 1 + x \log_e(1 + y) + \frac{1}{2!}\{x \log_e(1 + y)\}^2 + \ldots \to \infty. \quad (4)$$

Expanding $(1 + y)^x$ by the binomial theorem,

$$(1 + y)^x = 1 + xy + \frac{x(x-1)}{2!}y^2 + \frac{x(x-1)(x-2)}{3!}y^3 + \ldots$$

$$= 1 + x\{y + \frac{(-1)}{2!}y^2 + \frac{(-1)(-2)}{3!}y^3 + \ldots\}$$

$$+ \text{ terms in } x^2 \text{ and higher powers of } x. \quad . \quad . \quad (5)$$

Now the R.H. sides of (4) and (5) are the expansions of the same function and hence must be identical. Therefore, comparing coefficients of x on the R.H. sides of (4) and (5),

$$\log_e(1 + y) = y + \frac{(-1)}{2!}y^2 + \frac{(-1)(-2)}{3!}y^3 + \ldots \to \infty;$$

$$\boldsymbol{\log_e(1 + y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots \to \infty.}$$

This series is known as the *first logarithmic series* and has been shown to be convergent for $-1 < y \leqslant 1$ in a previous example.

Replacing y by $(-y)$ in this series, the *second logarithmic series* is obtained, viz.

$$\boldsymbol{\log_e(1 - y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \ldots \to \infty.}$$

If u_n be the nth term of this series, then

$$u_n = -\frac{y^n}{n}, \quad u_{n+1} = -\frac{y^{n+1}}{n+1} \quad \text{and} \quad \frac{u_{n+1}}{u_n} = y \cdot \frac{n}{n+1}.$$

$$\text{Hence } \left|\operatorname*{Lt}_{n\to\infty} \frac{u_{n+1}}{u_n}\right| = \left|\operatorname*{Lt}_{n\to\infty} \frac{n}{n+1}y\right| = \operatorname*{Lt}_{n\to\infty}\left|\frac{1}{1 + 1/n}y\right| = |y|.$$

Thus the series is convergent for $|y| < 1$, i.e. $-1 < y < 1$.

When $y = +1$ the series becomes $-1 - \frac{1}{2} - \frac{1}{3} \ldots \to \infty$

$$= -[1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots \to \infty].$$

The series inside the brackets is known to be divergent, and therefore the series is divergent for $y = +1$.

When $y = -1$ the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \to \infty$, which is known to be convergent. Hence the second logarithmic series is convergent for $-1 \leqslant y < 1$.

17. Theorem.

To find a series in ascending powers of y for $\log_e \frac{1+y}{1-y}$.

$$\log_e \frac{1+y}{1-y} = \log_e (1+y) - \log_e (1-y)$$

$$= \left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots\right) - \left(-y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \ldots\right)$$

$$= 2y + 2y^3/3 + 2y^5/5 + \ldots$$

$$= 2\{y + \frac{y^3}{3} + \frac{y^5}{5} + \ldots \to \infty\}.$$

Since this series is derived from the expansions of $\log_e (1+y)$ and $\log_e (1-y)$, it is valid for their common range of convergency, i.e. it is convergent for $-1 < y < 1$. This series is known as the *third logarithmic series.*

If y be replaced by $1/n$ in this third logarithmic series,

$$\log_e \frac{1+1/n}{1-1/n} = 2\left\{\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \ldots\right\} \text{ for } -1 < \frac{1}{n} < 1,$$

$$\text{i.e. } \log_e \frac{n+1}{n-1} = 2\left\{\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \ldots \to \infty\right\}.$$

This is the *fourth logarithmic series* and is valid for $-1 < 1/n < 1$, i.e. $-n < 1$ and $1 < n$ taken together, i.e. $n > -1$ and $n > 1$,

$$\text{i.e.} \quad n > 1.$$

If y be given the value $1/(2n+1)$ in the third logarithmic series,

$$\log_e \frac{1 + \frac{1}{2n+1}}{1 - \frac{1}{2n+1}} = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\right\}$$

$$\text{for } -1 < \frac{1}{2n+1} < 1,$$

$$\text{i.e.} \quad \log_e \frac{2n+2}{2n} = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\right\}$$

$$\text{for } (-2n-1) < 1 \text{ or } 1 < (2n+1),$$

$$\text{i.e.} \quad \log_e \frac{n+1}{n} = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\right\}$$

21. Theorem.

To find the expansions of $\sinh(x \pm y)$, $\cosh(x \pm y)$ and $\tanh(x \pm y)$.

By definition,

$$\begin{aligned}\sinh(x+y) &= \tfrac{1}{2}\{e^{x+y} - e^{-(x+y)}\} = \tfrac{1}{2}\{e^x \cdot e^y - e^{-x} \cdot e^{-y}\}\\ &= \tfrac{1}{2}\{(\cosh x + \sinh x)(\cosh y + \sinh y) - (\cosh x - \sinh x)\\ &\qquad \times (\cosh y - \sinh y)\}\\ &= \tfrac{1}{2}\{2\sinh x \cosh y + 2\cosh x \sinh y\}\\ &= \sinh x \cosh y + \cosh x \sinh y.\end{aligned}$$

Replacing y by $(-y)$ in this result,

$$\begin{aligned}\sinh(x-y) &= \sinh x \cosh(-y) + \cosh x \sinh(-y)\\ &= \sinh x \cosh y - \cosh x \sinh y.\end{aligned}$$

Similarly,

$$\begin{aligned}\cosh(x+y) &= \tfrac{1}{2}\{e^{x+y} + e^{-(x+y)}\} = \tfrac{1}{2}\{e^x \cdot e^y + e^{-x} \cdot e^{-y}\}\\ &= \tfrac{1}{2}\{(\cosh x + \sinh x)(\cosh y + \sinh y) + (\cosh x - \sinh x)\\ &\qquad \times (\cosh y - \sinh y)\}\\ &= \tfrac{1}{2}\{2\cosh x \cosh y + 2\sinh x \sinh y\}\\ &= \cosh x \cosh y + \sinh x \sinh y.\end{aligned}$$

Replacing y by $(-y)$ in this,

$$\begin{aligned}\cosh(x-y) &= \cosh x \cosh(-y) + \sinh x \sinh(-y)\\ &= \cosh x \cosh y - \sinh x \sinh y.\end{aligned}$$

$$\begin{aligned}\tanh(x+y) &= \frac{\sinh(x+y)}{\cosh(x+y)} = \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y}\\ &= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \quad \text{(dividing numerator and denominator by } \cosh x \cosh y\text{).}\end{aligned}$$

Replacing y by $(-y)$ in this result,

$$\begin{aligned}\tanh(x-y) &= \frac{\tanh x + \tanh(-y)}{1 + \tanh x \tanh(-y)}\\ &= \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}.\end{aligned}$$

The results (13), (14), (15), and (16) should also be memorized in the reverse form, namely

$$\left.\begin{aligned}
2\sinh x\cosh y &= \sinh(x+y) + \sinh(x-y)\\
2\cosh x\sinh y &= \sinh(x+y) - \sinh(x-y)\\
2\cosh x\cosh y &= \cosh(x+y) + \cosh(x-y)\\
2\sinh x\sinh y &= \cosh(x+y) - \cosh(x-y)
\end{aligned}\right\}.$$

24. Theorem.

To find the series for $\cosh x$ and $\sinh x$ in ascending powers of x.

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}) = \tfrac{1}{2}\left\{\begin{array}{l}(1 + x + x^2/2! + x^3/3! + \dots)\\ -(1 - x + x^2/2! - x^3/3! + \dots)\end{array}\right\}$$

$$= \tfrac{1}{2}\left\{2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \dots\right\}$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \to \infty.$$

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}) = \tfrac{1}{2}\left\{\begin{array}{l}(1 + x + x^2/2! + x^3/3! + \dots)\\ +(1 - x + x^2/2! - x^3/3! + \dots)\end{array}\right\}$$

$$= \tfrac{1}{2}\left\{2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \dots\right\}$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \to \infty.$$

Both these series, being derived from the series for e^x, can be seen to be convergent for all values of x.

25. Theorem.

To find the solutions of the equation $a\cosh x + b\sinh x = c$, where a, b, c are constants.

Using the definitions of $\cosh x$ and $\sinh x$, the equation becomes

$$\frac{a}{2}(e^x + e^{-x}) + \frac{b}{2}(e^x - e^{-x}) = c,$$

i.e. $$a(e^{2x} + 1) + b(e^{2x} - 1) = 2ce^x.$$

$$\therefore\quad (a+b)e^{2x} - 2ce^x + (a-b) = 0.$$

This is a quadratic in e^x and can be solved in the usual manner for e^x, from which the values of x can be determined.

Example 11.—Solve the equation $3\cosh x + 2\sinh x = 3$.

Using the definitions for $\sinh x$ and $\cosh x$, the equation becomes

$$\tfrac{3}{2}(e^x + e^{-x}) + \tfrac{2}{2}(e^x - e^{-x}) = 3,$$

i.e. $$3(e^{2x} + 1) + 2(e^{2x} - 1) = 6e^x,$$

i.e. $$5e^{2x} - 6e^x + 1 = 0.$$

$$\therefore \quad (5e^x - 1)(e^x - 1) = 0.$$

$$\therefore \quad e^x = \tfrac{1}{5} \text{ or } 1.$$

$$\therefore \quad \left.\begin{aligned} x &= \log_e \tfrac{1}{5} = -\log_e 5 = -1{\cdot}60944 \\ \text{or} \quad x &= \log_e 1 = 0 \end{aligned}\right\}.$$

26. Inverse Hyperbolic Functions.

Corresponding to the inverse trigonometric functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, etc., representing the angle between $-\pi/2$ and $+\pi/2$ whose sine has the value x, the angle between 0 and π whose cosine has the value x, and the angle between $-\pi/2$ and $+\pi/2$ whose tangent has the value x, etc., respectively, there are inverse hyperbolic functions. These are $\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$, $\operatorname{sech}^{-1} x$, $\operatorname{cosech}^{-1} x$, and $\coth^{-1} x$, and represent the angles (in radians) whose hyperbolic sine, cosine, tangent, secant, cosecant, and cotangent have the value x respectively. There are no limits attached to the size of the angle.

27. Theorem.

To find the values of $\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$, etc., in terms of x.

(i) Let $\sinh^{-1} x = y$.

Then $$x = \sinh y = \tfrac{1}{2}(e^y - e^{-y}).$$

$$\therefore \quad 2xe^y = e^{2y} - 1, \text{ i.e. } e^{2y} - 2xe^y - 1 = 0;$$

$$\therefore \quad e^y = x \pm \sqrt{x^2 + 1}.$$

But e^y is essentially positive, therefore the positive sign only can be used.

Hence $$e^y = x + \sqrt{x^2 + 1}$$

and $$y = \sinh^{-1} x = \log_e (x + \sqrt{x^2 + 1}).$$

(ii) Let $\cosh^{-1} x = y$, then $x = \cosh y = \tfrac{1}{2}(e^y + e^{-y})$,

i.e. $$2xe^y = e^{2y} + 1,$$

i.e. $$e^{2y} - 2xe^y + 1 = 0.$$

$$\therefore \quad e^y = x \pm \sqrt{x^2 - 1} \text{ (both signs admissible)}$$

and $$y = \cosh^{-1} x = \log_e (x \pm \sqrt{x^2 - 1}).$$

The reason for the ambiguity in sign is that the graph of $y = \cosh^{-1} x$ is symmetrical about OX.

(iii) Let $\tanh^{-1} x = y$.

Then $$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1}.$$

$$\therefore \quad x(e^{2y} + 1) = e^{2y} - 1.$$

$$\therefore \quad (1 - x)e^{2y} = 1 + x,$$

i.e. $$e^{2y} = (1 + x)/(1 - x) \qquad (x^2 < 1 \text{ since } e^{2y} \text{ is positive}).$$

$$\therefore \quad 2y = \log_e \frac{1 + x}{1 - x},$$

i.e. $$y = \tanh^{-1} x = \tfrac{1}{2} \log_e \frac{1 + x}{1 - x} \; (x^2 < 1).$$

The results for $\operatorname{cosech}^{-1} x$, $\operatorname{sech}^{-1} x$, $\coth^{-1} x$ can be obtained from these results by replacing x by $1/x$ in the values for $\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$ respectively, since $\operatorname{cosech} y = 1/\sinh y$, etc.

Example 12.—Find the value of $\sinh^{-1} 3/4$.

$$\sinh^{-1} x = \log_e (x + \sqrt{x^2 + 1}).$$

$$\begin{aligned} \therefore \quad \sinh^{-1} 3/4 &= \log_e (\tfrac{3}{4} + \sqrt{\tfrac{9}{16} + 1} \\ &= \log_e (\tfrac{3}{4} + \tfrac{5}{4}) = \log_e 2 \\ &= 0{\cdot}69315 \text{ to five decimal places.} \end{aligned}$$

COMPLEX NUMBERS

28. Any number which can be found as a point on the *number scale* extending from $-\infty$ to $+\infty$ (an indefinitely long straight line with some fixed point O used as the zero mark, and a convenient scale to represent one unit) is known as a *real number*, and these are subdivided into *rational* and *irrational* numbers. A rational number is any number that can be expressed in the form p/q, where p and q are integers, and an irrational number is any real number that cannot be expressed in this form.

The square root of -1 is denoted by the symbol i ($i = \sqrt{-1}$). The value of this quantity cannot be determined as a real number, and therefore the product of any real number and i, which also cannot be placed on the number scale, is known as an *imaginary number*. From this it can be seen that no real number, except zero, can be

equal to an imaginary number. [Zero is a neutral number and can be taken to be real or imaginary.]

A *complex number* is the sum of a real number and an imaginary number. Thus, if a and b be real numbers, a complex number will be represented by $a + ib$, which is the standard form of representation. It will have been found that complex numbers have arisen as the roots of certain quadratic equations and, therefore, all the processes of algebra are applicable to complex numbers.

It is to be noted that $i^2 = -1$, $i^3 = -i$, $i^4 = +1$, $i^5 = i$, and these values occur in cycles for higher powers of i.

29. Theorem.

If two complex numbers are equal then their real portions are equal, and their imaginary portions are also separately equal.

Let $a + ib$ and $c + id$ be the two complex numbers.

Since
$$a + ib = c + id,$$
$$\therefore \quad a - c = i(d - b),$$

i.e. since a real number cannot equal an imaginary number unless each be zero it must follow that

$$a - c = 0 \text{ and } d - b = 0,$$

i.e.
$$a = c \text{ and } d = b.$$

This theorem is applied to many problems on complex numbers.

Note.—If a complex number be given in the form $a + ib$, it is understood that a and b are real.

Definition.—When the two complex numbers $(x + iy)$ and $(x - iy)$ are multiplied together, the result is $x^2 + y^2$, which is a real number. The quantity $x - iy$ is known as the *conjugate complex* of $(x + iy)$ and vice versa.

If it be required to simplify a fraction having a complex number in the denominator, both numerator and denominator of the fraction must be multiplied by the conjugate complex of the denominator.

30. Theorem.

To express in the form of a complex number the sum, difference, product and quotient of two complex numbers.

Let $(a + ib)$ and $(c + id)$ be the two complex numbers,

Their sum $= (a + ib) + (c + id) = (a + c) + i(b + d).$

Their difference $= (a + ib) - (c + id) = a - c + i(b - d).$

Their product $= (a + ib)(c + id) = ac + i^2bd + i(ad + bc)$

$= ac - bd + i(ad + bc).$

Their quotient $= \dfrac{a + ib}{c + id} = \dfrac{(a + ib)(c - id)}{(c + id)(c - id)}$

$$= \frac{ac - i^2bd + i(bc - ad)}{c^2 - i^2d^2} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}.$$

Definition.—If $a + ib$ be expressed in the form $r(\cos\theta + i\sin\theta)$, where r is positive and $0 \leqslant \theta \leqslant 360°$, then r is known as the *modulus* of the complex number, and θ as the *argument* or *amplitude* of the complex number.

31. Theorem.

To express the modulus and argument of the complex number $a + ib$ in terms of a and b.

Let r be the modulus (always +ve), and θ the argument ($0 \leqslant \theta \leqslant 360°$).

Then $$a + ib = r(\cos\theta + i\sin\theta).$$

Equating real and imaginaries,

$$a = r\cos\theta, \quad \dots \dots \quad (17)$$

$$b = r\sin\theta. \quad \dots \dots \quad (18)$$

$(17)^2 + (18)^2$ $\qquad a^2 + b^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2.$

$$\left.\begin{aligned} \therefore\ r &= \sqrt{(a^2 + b^2)} \\ \tan\theta &= b/a \end{aligned}\right\}.$$

$(18) \div (17)$

N.B.—The quadrant in which θ lies will always be determined from equations (17) and (18).

It is worth noting that

$$1 = 1(\cos 0° + i\sin 0°);$$

$$i = 1(\cos 90° + i\sin 90°);$$

$$-1 = i^2 = 1(\cos 180° + i\sin 180°);$$

$$-i = i^3 = 1(\cos 270° + i\sin 270°).$$

Example 13.—Express (*a*) $(2 + 3i)(3 - 2i)$; (*b*) $\dfrac{1}{(2+i)^2} - \dfrac{1}{(2-i)^2}$ in the form $a + ib$, and find their moduli and arguments.

(*a*) $(2 + 3i)(3 - 2i) = 6 - 6i^2 + i(9 - 4) = 6 + 6 + i\,.\,5 = 12 + 5i.$

Let r be its modulus and θ its argument.

Then $$12 + 5i = r(\cos\theta + i\sin\theta).$$

Equating real and imaginaries,

$$12 = r\cos\theta. \quad \ldots \quad \text{(i)}$$

$$5 = r\sin\theta \quad \ldots \quad \text{(ii)}$$

$\text{(i)}^2 + \text{(ii)}^2$ $$r^2 = 5^2 + 12^2 = 169.$$

$$\therefore\ r = 13.$$

From (i) and (ii), θ lies in the first quadrant, since $\cos\theta$ and $\sin\theta$ are both positive, and $\tan\theta = 5/12$, i.e. $\cot\theta = 2{\cdot}4$, $\therefore\ \theta = 22^\circ\ 37'$.

(*b*) $$\frac{1}{(2+i)^2} - \frac{1}{(2-i)^2} = \frac{(2-i)^2 - (2+i)^2}{\{(2+i)(2-i)\}^2} = -\frac{8i}{5^2} = -\frac{8i}{25} = 0 - 0{\cdot}32i.$$

Now $$-i = 1(\cos 270^\circ + i\sin 270^\circ),$$

$$\therefore\ -0{\cdot}32i = 0{\cdot}32(\cos 270^\circ + i\sin 270^\circ).$$

Therefore $$\left.\begin{array}{r}\text{the modulus is } 0{\cdot}32\\ \text{and the argument } 270^\circ\end{array}\right\}.$$

32. Theorem.

To find the modulus and argument of the product and quotient of two complex numbers.

Let $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$ be the complex numbers, sometimes denoted by $r_1[\theta_1]$ and $r_2[\theta_2]$ respectively, or (r_1, θ_1), (r_2, θ_2).

The product $= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

$= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$,

i.e. a complex number of *modulus* r_1r_2 *and argument* $(\theta_1 + \theta_2)$.

$$\text{The quotient} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\left[\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right]$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)],$$

i.e. a complex number of *modulus* r_1/r_2 *and argument* $(\theta_1 - \theta_2)$.

Thus *the modulus of the product of two complex numbers is the product of their moduli*, and *the argument of the product is the sum of their arguments*. Also *the modulus of the quotient of two complex numbers is the quotient of their moduli* and *the argument of their quotient is the difference* (argument of numerator diminished by that of the denominator) *of their arguments*.

38. Demoivre's Theorem.

This states that, for all rational values of n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

There are three separate cases to be considered in the proof, (i) n a positive integer, (ii) n a negative integer, (iii) n fractional.

Case (i). n is a positive integer.

$$(\cos x + i\sin x)(\cos y + i\sin y)$$
$$= (\cos x\cos y - \sin x\sin y) + i(\sin x\cos y + \cos x\sin y)$$
$$= \cos(x+y) + i\sin(x+y). \quad \dots\dots\dots\dots \quad (19)$$

$\therefore\ (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$
(using (19) with $x = \theta_1$, $y = \theta_2$).

$\therefore\ (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$
$$= \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}\{\cos\theta_3 + i\sin\theta_3\}$$
$$= \cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)$$
(using (19) with $x = \theta_1 + \theta_2$, $y = \theta_3$).

Continuing this process it can be seen that, if n be a positive integer,

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\dots(\cos\theta_n + i\sin\theta_n)$
$$= \cos(\theta_1+\theta_2+\theta_3+\dots+\theta_n) + i\sin(\theta_1+\theta_2+\theta_3+\dots+\theta_n).$$

Using $\theta_1 = \theta_2 = \theta_3 = \dots = \theta_n = \theta$ in this result, it becomes

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Case (ii). n is a negative integer.

Let $n = -m$, $\therefore$ m is a positive integer.

Then $$(\cos\theta + i\sin\theta)^n = \frac{1}{(\cos\theta + i\sin\theta)^{-n}} = \frac{1}{(\cos\theta + i\sin\theta)^m}$$
$$= \frac{1}{\cos m\theta + i\sin m\theta} \text{ (using case (i))}$$
$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$
$$= \frac{\cos(-m\theta) + i\sin(-m\theta)}{\cos^2 m\theta + \sin^2 m\theta}$$
$$= \cos n\theta + i\sin n\theta.$$

Case (iii). n is fractional.

Let $n = p/q$, where p and q are integers.

Then $\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos\theta + i\sin\theta$ (using cases (i) and (ii)).

It follows that $\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)$ is one of the qth roots of $\cos\theta + i\sin\theta$, and this statement can be written

$$\cos\frac{\theta}{q} + i\sin\frac{\theta}{q} = (\cos\theta + i\sin\theta)^{1/q}.$$

$$\therefore \quad \left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p = \{(\cos\theta + i\sin\theta)^{1/q}\}^p$$
$$= (\cos\theta + i\sin\theta)^{p/q}.$$

Using cases (i) and (ii),

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p = \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta.$$

$$\therefore \quad (\cos\theta + i\sin\theta)^{p/q} = \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta,$$

i.e. $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Thus for all rational values of n

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

34. Theorem.

To find the expansions of $\cos n\theta$ and $\sin n\theta$ in terms of sines and cosines of the angle θ, where n is a positive integer.

Using Demoivre's theorem,

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n$$

$$= \cos^n\theta + i\,{}^nC_1 \cos^{n-1}\theta\sin\theta + i^2\,{}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots$$

$$+ i^r\,{}^nC_r\cos^{n-r}\theta\sin^r\theta + \ldots$$ (where ${}^nC_1$, ${}^nC_2$, etc., are the coefficients in the binomial expansion when n is a +ve integer, see § 10)

$$= \cos^n\theta + i\,{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta - i\,{}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots$$

Equating real and imaginary quantities in this,

$$\left.\begin{aligned} \cos n\theta &= \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots \\ \sin n\theta &= {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \ldots \end{aligned}\right\}$$

Example 14.—Find $\sin 3\theta$ in terms of $\sin\theta$, and $\cos 3\theta$ in terms of $\cos\theta$, using Demoivre's theorem.

$$\begin{aligned} \cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta. \end{aligned}$$

Equating real and imaginaries,

$$\begin{aligned} \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta; \\ \sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta \\ &= 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta. \end{aligned}$$

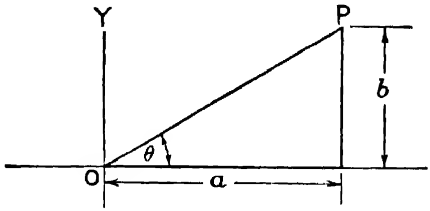
35. The Argand Diagram.

This diagram is a means of representing a complex number on graph paper. It consists of the usual rectangular axes OX, OY, with the same scales for units along both OX and OY. All real numbers are measured along or parallel to OX, e.g. the number 2 will be represented by a point two units along OX in the positive direction, and imaginary numbers are measured along or parallel to OY, e.g. $3i$ will be represented by a point on OY, three units from O in the positive direction.

Thus the complex number $2 + 3i$ will be represented by a point P in the first quadrant (or the line OP) which is 2 units from OY and 3 units from OX. Similarly a complex number $-2 - 3i$ will be a

point in the third quadrant which is 2 units from OY and 3 units from OX.

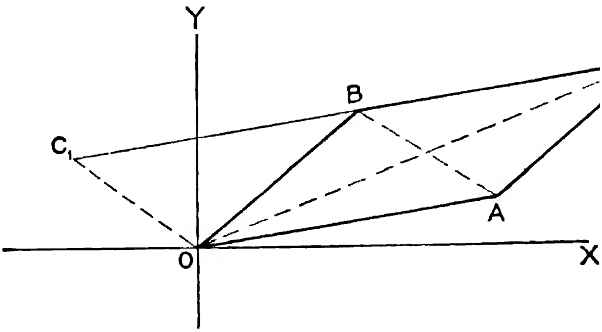
In general, the complex number $a + ib$ will be represented by a point a units from OY and b units from OX, with the usual graphical conventions applying to signs.



In the diagram, P (or OP) represents the complex number $a + ib$. Let the length of OP be r, and the angle POX be θ.

Then $$a = r\cos\theta \text{ and } b = r\sin\theta.$$
$$\therefore \quad a + ib = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta).$$

Hence r is the modulus of the complex number and θ is its argument, i.e. the complex number whose modulus is r and argument θ can be represented on the Argand diagram by a point P (or the line OP), where OP is of length r and $\angle$POX $= \theta$.



36. Addition and Subtraction on the Argand diagram.

Let $a + ib$ and $c + id$ be two complex numbers represented by OA and OB respectively on the Argand diagram. Complete the parallelogram OACB. Since AC is equal and parallel to OB, its projection on OX is equal to the projection of OB on OX which equals c.

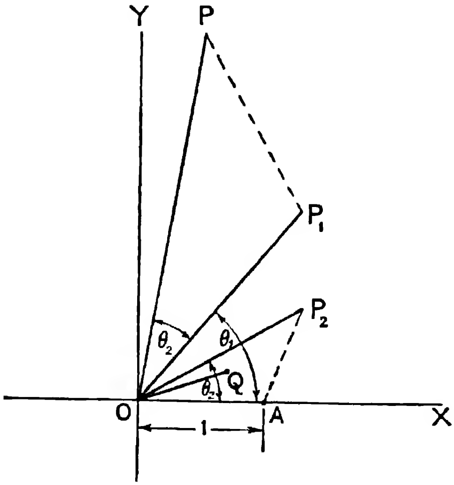
The projection of OA on OX is a. Hence from the diagram it can be seen that the projection of OC on the x-axis is $(a + c)$, and similarly the projection of OC on the y-axis is $(b + d)$. Thus the line OC represents the complex number $(a + c) + i(b + d)$, which is the sum of the complex numbers represented by OA and OB.

If CB be produced to C_1, so that $BC_1 = CB$, and the parallelogram $OABC_1$ completed, it can be readily seen that OC_1 represents the complex number $(c - a) + i(d - b)$, which is the difference of the complex numbers represented by OB and OA.

Hence, it can be seen that, on the Argand diagram, the addition and subtraction of complex numbers follow the vector laws.

37. Multiplication and division of complex numbers on the Argand diagram.

Consider the two complex numbers (r_1, θ_1), (r_2, θ_2). It has been proved that their product is a complex number whose modulus is r_1r_2 and whose argument is $(\theta_1 + \theta_2)$, and their quotient has modulus r_1/r_2 and argument $\theta_1 - \theta_2$. Hence, if OP_1, OP_2 represent the two



complex numbers on the Argand diagram, their product will be represented by OP, where $OP = OP_1 \times OP_2$ and $\angle POX = \angle P_1OX + \angle P_2OX$. Also their quotient will be represented by OQ, where $OQ = OP_1/OP_2$ and $\angle QOX = \angle P_1OX - \angle P_2OX$.

OP can be found geometrically as follows:

Take A on OX so that $OA = 1$ unit. Join AP_2. On OP_1 construct

Let z be represented by OA and a by OB on the Argand diagram. Complete the parallelogram OBCA and the parallelogram OBAC$_1$, as shown in the diagram.

Then length OC represents $|z + a|$.

Length $OC_1 = AB$ (by geometry) and represents $|z - a|$.

The lines OA and OB represent $|z|$ and $|a|$ respectively.

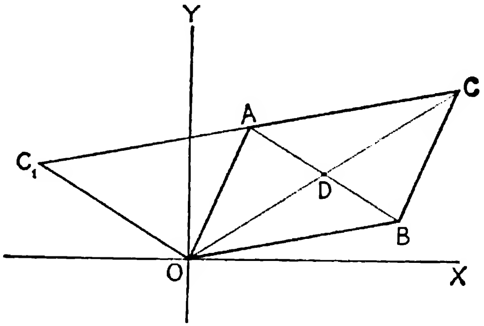
Let D be the mid-point of AB, i.e. the intersection of diagonals AB and OC.

From triangle OAB, $2OD^2 + 2AD^2 = OA^2 + OB^2$,

i.e. $$\tfrac{1}{2}OC^2 + \tfrac{1}{2}AB^2 = OA^2 + OB^2, \quad \ldots \ldots \quad (i)$$

i.e. $$OC^2 + AB^2 = 2(OA^2 + OB^2).$$

$$\therefore |z + a|^2 + |z - a|^2 = 2\{|z|^2 + |a|^2\}.$$



38. The exponential values of sin θ and cos θ.

It will be assumed that

$$e^z = \underset{n \to \infty}{\text{Lt}} \left(1 + \frac{z}{n}\right)^n = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \to \infty$$

holds true for all values of z including complex numbers.

Let $\quad z = i\theta$, then $e^{i\theta} = \underset{n \to \infty}{\text{Lt}} \left(1 + \frac{i\theta}{n}\right)^n$.

Now as $n \to \infty$, $\tan \frac{\theta}{n} \to \frac{\theta}{n}$, and using this,

$$e^{i\theta} = \underset{n \to \infty}{\text{Lt}} (1 + i \tan \theta/n)^n$$

$$= \underset{n \to \infty}{\text{Lt}} \left(\frac{\cos \theta/n + i \sin \theta/n}{\cos \theta/n}\right)^n$$

$$= \underset{n \to \infty}{\text{Lt}} \frac{\cos \theta + i \sin \theta}{\cos^n \theta/n} \text{(using Demoivre's theorem).}$$

But $$\underset{n \to \infty}{\text{Lt}} (\cos^n \theta/n) = 1.$$

$$\therefore e^{i\theta} = \cos \theta + i \sin \theta. \quad \ldots \ldots \quad (20)$$

Replacing θ by $(-\theta)$ in (20),

$$\begin{aligned} e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta - i\sin\theta. \qquad (21) \end{aligned}$$

From (20) × (21) $\qquad e^0 = \cos^2\theta + \sin^2\theta,$

i.e. $\qquad \cos^2\theta + \sin^2\theta = 1,$

which is the fundamental identity in trigonometry.

$$\left.\begin{aligned} &\text{From } (20) + (21) \qquad \cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}); \\ &(20) - (21) \qquad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}). \end{aligned}\right\}$$

39. Theorem.

To find the series for $\sin\theta$ and $\cos\theta$ in ascending powers of θ.

$$\begin{aligned} \cos\theta &= \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) \\ &= \tfrac{1}{2}\left[\begin{aligned} &\left(1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots\right) \\ &+ \left(1 - i\theta + \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} - \ldots\right) \end{aligned}\right] \\ &= \tfrac{1}{2}\left[2 + 2\frac{i^2\theta^2}{2!} + 2\frac{i^4\theta^4}{4!} + \ldots\right] \\ &= 1 + \frac{i^2\theta^2}{2!} + \frac{i^4\theta^4}{4!} + \ldots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots \end{aligned}$$

$$\begin{aligned} \sin\theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \\ &= \frac{1}{2i}\left[\begin{aligned} &\left(1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots\right) \\ &-\left(1 - i\theta + \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} - \ldots\right) \end{aligned}\right] \\ &= \frac{1}{2i}\left[2i\theta + 2\frac{i^3\theta^3}{3!} + 2\frac{i^5\theta^5}{5!} + \ldots\right] \\ &= \theta + \frac{i^2\theta^3}{3!} + \frac{i^4\theta^5}{5!} + \ldots \\ &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots \end{aligned}$$

40. Theorem.

To find $\cos^n\theta$ and $\sin^n\theta$ in terms of the sines and cosines of the multiple angles, where n is a positive integer.

$$\cos^n\theta = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^n = \frac{1}{2^n}(e^{i\theta}+e^{-i\theta})^n.$$

$$\therefore\ 2^n\cos^n\theta = e^{in\theta} + {}^nC_1 e^{i(n-2)\theta} + {}^nC_2 e^{i(n-4)\theta} + \dots$$
$$+ {}^nC_2 e^{-i(n-4)\theta} + {}^nC_1 e^{-i(n-2)\theta} + e^{-in\theta}$$
$$= (e^{in\theta}+e^{-in\theta}) + {}^nC_1\{e^{i(n-2)\theta}+e^{-i(n-2)\theta}\}$$
$$+ {}^nC_2\{e^{i(n-4)\theta}+e^{-i(n-4)\theta}\} + \dots$$
$$= 2\cos n\theta + 2\,.\,{}^nC_1\cos(n-2)\theta$$
$$+ 2\,.\,{}^nC_2\cos(n-4)\theta + \dots$$

$$\therefore \cos^n\theta = \frac{1}{2^{n-1}}\{\cos n\theta + {}^nC_1\cos(n-2)\theta + {}^nC_2\cos(n-4)\theta + \dots\}.$$

$$(2i\sin\theta)^n = (e^{i\theta}-e^{-i\theta})^n$$
$$= e^{in\theta} - {}^nC_1 e^{i(n-2)\theta} + {}^nC_2 e^{i(n-4)\theta} + \dots$$
$$+ (-1)^{n-2}\,{}^nC_2 e^{-i(n-4)\theta} + (-1)^{n-1}\,{}^nC_1 e^{-i(n-2)\theta} + (-1)^n e^{-in\theta}$$
$$= \{e^{in\theta} + (-1)^n e^{-in\theta}\} - {}^nC_1\{e^{i(n-2)\theta} + (-1)^n e^{-i(n-2)\theta}\}$$
$$+ {}^nC_2\{e^{i(n-4)\theta} - (-1)^n e^{i(n-4)\theta}\} + \dots$$

There are two cases to be considered as follows:

Case (i). When n is even,

$2^n i^n \sin^n\theta$

$$= 2\cos n\theta - 2\,.\,{}^nC_1\cos(n-2)\theta + 2\,.\,{}^nC_2\cos(n-4)\theta - \dots,$$

i.e. $\sin^n\theta$

$$= \frac{1}{(-1)^{n/2}2^{n-1}}[\cos n\theta - {}^nC_1\cos(n-2)\theta + {}^nC_2\cos(n-4)\theta - \dots].$$

Case (ii). When n is odd,

$2^n i(i^{n-1})\sin^n\theta$

$$= 2i\sin n\theta - 2i\,{}^nC_1\sin(n-2)\theta + 2i\,{}^nC_2\sin(n-4)\theta - \dots$$

$\therefore\ \sin^n\theta$

$$= \frac{1}{(-1)^{\frac{n-1}{2}}\,2^{n-1}}[\sin n\theta - {}^nC_1\sin(n-2)\theta + {}^nC_2\sin(n-4)\theta - \dots].$$

Example 18 (L.U.).—Express $\cos 7\theta$ as a polynomial in $\cos\theta$, and prove that, if $x = 2\cos\theta$, then

$$\frac{1+\cos 7\theta}{1+\cos\theta} = (x^3 - x^2 - 2x + 1)^2.$$

Using $\cos\theta + i\sin\theta = e^{i\theta}$,

$$\begin{aligned}
\cos 7\theta + i\sin 7\theta = e^{i7\theta} &= (e^{i\theta})^7 \\
&= (\cos\theta + i\sin\theta)^7 \\
&= \cos^7\theta + 7i\cos^6\theta\sin\theta + 21i^2\cos^5\theta\sin^2\theta \\
&\quad + 35i^3\cos^4\theta\sin^3\theta + 35i^4\cos^3\theta\sin^4\theta + 21i^5\cos^2\theta\sin^5\theta \\
&\quad + 7i^6\cos\theta\sin^6\theta + i^7\sin^7\theta \\
&= \cos^7\theta + 7i\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - 35i\cos^4\theta\sin^3\theta \\
&\quad + 35\cos^3\theta\sin^4\theta + 21i\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - i\sin^7\theta.
\end{aligned}$$

Equating real quantities,

$$\begin{aligned}
\cos 7\theta &= \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta \\
&= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-\cos^2\theta)^2 \\
&\quad - 7\cos\theta(1-\cos^2\theta)^3 \\
&= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta - 7\cos\theta \\
&\quad + 21\cos^3\theta - 21\cos^5\theta + 7\cos^7\theta \\
&= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta.
\end{aligned}$$

$$\begin{aligned}
\frac{1+\cos 7\theta}{1+\cos\theta} = \frac{2\cos^2\frac{7\theta}{2}}{2\cos^2\frac{\theta}{2}} &= \left\{\frac{64\cos^7\frac{\theta}{2} - 112\cos^5\frac{\theta}{2} + 56\cos^3\frac{\theta}{2} - 7\cos\frac{\theta}{2}}{\cos\theta/2}\right\}^2 \\
&= (64\cos^6\frac{\theta}{2} - 112\cos^4\frac{\theta}{2} + 56\cos^2\frac{\theta}{2} - 7)^2.
\end{aligned}$$

But $\cos^2\theta/2 = \frac{1}{2}(1+\cos\theta) = \frac{1}{2}(1+\frac{1}{2}x) = \frac{1}{4}(2+x)$.

$$\begin{aligned}
\therefore\ \frac{1+\cos 7\theta}{1+\cos\theta} &= \left\{64\times\frac{1}{4^3}(2+x)^3 - 112\times\frac{1}{16}(2+x)^2 + 56\times\frac{1}{4}(2+x) - 7\right\}^2 \\
&= \{(2+x)^3 - 7(2+x)^2 + 14(2+x) - 7\}^2 \\
&= \{8 + 12x + 6x^2 + x^3 - 28 - 28x - 7x^2 + 28 + 14x - 7\}^2 \\
&= \{x^3 - x^2 - 2x + 1\}^2.
\end{aligned}$$

41. Connection between the trigonometric and hyperbolic functions.

From previous results,

$$\cos i\theta = \tfrac{1}{2}\left\{e^{i(i\theta)} + e^{-i(i\theta)}\right\} = \tfrac{1}{2}\left\{e^{-\theta} + e^{\theta}\right\} = \cosh\theta.$$

$$\sin i\theta = \frac{1}{2i}\left\{e^{i(i\theta)} - e^{-i(i\theta)}\right\} = \frac{1}{2i}\left\{e^{-\theta} - e^{\theta}\right\}$$

$$= \frac{-1}{2i}\left\{e^{\theta} - e^{-\theta}\right\} = i\,.\,\tfrac{1}{2}\left\{e^{\theta} - e^{-\theta}\right\} = i\sinh\theta.$$

$$\tan i\theta = \frac{\sin i\theta}{\cos i\theta} = \frac{i\sinh\theta}{\cosh\theta} = i\tanh\theta.$$

$$\cosh i\theta = \tfrac{1}{2}\left\{e^{i\theta} + e^{-(i\theta)}\right\} = \cos\theta.$$

$$\sinh i\theta = \tfrac{1}{2}\left\{e^{(i\theta)} - e^{-(i\theta)}\right\} = i \times \frac{1}{2i}\left\{e^{i\theta} - e^{-i\theta}\right\} = i\sin\theta.$$

$$\tanh i\theta = \frac{\sinh i\theta}{\cosh i\theta} = \frac{i\sin\theta}{\cos\theta} = i\tan\theta.$$

Example 19.—Prove that, if $\sin(x + iy)$ be expressed in the form $r[\cos\theta + i\sin\theta]$, then $r = \sqrt{\left(\dfrac{\cosh 2y - \cos 2x}{2}\right)}$ and $\tan\theta = \cot x\tanh y$, where x, y, r, θ are real.

Since $\qquad \sin(x + iy) = r[\cos\theta + i\sin\theta],$

$$r\cos\theta + ir\sin\theta = \sin x\cos iy + \cos x\sin iy$$
$$= \sin x\cosh y + i\cos x\sinh y.$$

Equating real and imaginary quantities,

$$r\cos\theta = \sin x\cosh y, \quad \ldots\ldots\ldots \quad \text{(i)}$$
$$r\sin\theta = \cos x\sinh y. \quad \ldots\ldots\ldots \quad \text{(ii)}$$

(i)^2 + (ii)^2 $\qquad r^2 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y$

$$= \frac{(1 - \cos 2x)}{2}\frac{(1 + \cosh 2y)}{2} + \frac{(1 + \cos 2x)}{2}\frac{(\cosh 2y - 1)}{2}$$

$$= \tfrac{1}{4}\left\{\begin{matrix}1 + \cosh 2y - \cos 2x - \cosh 2y\cos 2x \\ + \cosh 2y - 1 + \cos 2x\cosh 2y - \cos 2x\end{matrix}\right\}$$

$$= \tfrac{1}{4}\{2\cosh 2y - 2\cos 2x\}$$

$$= \tfrac{1}{2}\{\cosh 2y - \cos 2x\}.$$

$$\therefore\quad r = \sqrt{\left(\frac{\cosh 2y - \cos 2x}{2}\right)}.$$

From (ii) ÷ (i) $\qquad \tan\theta = \tanh y\cot x.$

Example 20.—Find in the form $a + ib$ *one* value of each of the following expressions:

$$(a) \qquad \log_e \frac{3-i}{3+i}; \qquad (b) \ \cos^{-1}\frac{3i}{4}.$$

(*a*) Let
$$\log_e \frac{3-i}{3+i} = x + iy.$$

$$\therefore \quad \frac{3-i}{3+i} = e^{x+iy},$$

i.e.
$$\frac{(3-i)^2}{(3+i)(3-i)} = e^x \cdot e^{iy},$$

i.e.
$$\frac{8-6i}{10} = e^x(\cos y + i \sin y).$$

Equating real and imaginary quantities,

$$e^x \cos y = 4/5, \quad \ldots \ldots \quad \text{(i)}$$

$$e^x \sin y = -3/5. \quad \ldots \ldots \quad \text{(ii)}$$

From $(\text{i})^2 + (\text{ii})^2$ $\qquad e^{2x} = 1, \ \therefore \ x = 0;$

and (i) and (ii) become $\qquad \cos y = 0{\cdot}8, \ \sin y = -0{\cdot}6.$

$\therefore$ angle y lies in the fourth quadrant, and one value of $y = -0{\cdot}6434$ radians.

$\therefore$ one value of $x + iy$ is $0 - i \,.\, 0{\cdot}6434$.

(*b*) Let
$$\cos^{-1}\frac{3i}{4} = x + iy, \quad \therefore \quad \frac{3i}{4} = \cos(x + iy),$$

i.e.
$$\frac{3i}{4} = \cos x \cos iy - \sin x \sin iy$$

$$= \cos x \cosh y - i \sin x \sinh y.$$

Equating real and imaginary quantities,

$$\cos x \cosh y = 0, \quad \ldots \ldots \quad \text{(iii)}$$

$$\sin x \sinh y = -3/4. \quad \ldots \ldots \quad \text{(iv)}$$

Now $\cosh y$ cannot be less than 1, $\therefore \cosh y \neq 0$.

$\therefore$ from (iii), $\cos x = 0$, and one value of x is $\pi/2$.

Using this in (iv), $\sinh y = -\frac{3}{4}$, and

$$y = \sinh^{-1}(-\tfrac{3}{4}) = \log_e\left\{-\frac{3}{4} + \sqrt{\left(\frac{9}{16} + 1\right)}\right\}$$

$$= \log_e \tfrac{1}{2} = -\log_e 2.$$

$$\therefore \text{ one value of } \cos^{-1}\frac{3i}{4} \text{ is } \frac{\pi}{2} - i \log_e 2.$$

Example 21 (L.U.).—If $z = x + iy = \tanh\left(u + \frac{i\pi}{4}\right)$, where u is real, find x and y in terms of u, and show that for all values of u the point z, on the Argand diagram, lies on the circle $x^2 + y^2 = 1$.

$$x + iy = \tanh\left(u + \frac{i\pi}{4}\right) = \frac{\tanh u + \tanh \frac{i\pi}{4}}{1 + \tanh u \tanh \frac{i\pi}{4}}$$

$$= \frac{\tanh u + i \tan \frac{\pi}{4}}{1 + i \tanh u \tan \frac{\pi}{4}} = \frac{\tanh u + i}{1 + i \tanh u}$$

$$= \frac{(\tanh u + i)(1 - i \tanh u)}{(1 + i \tanh u)(1 - i \tanh u)}$$

$$= \frac{\tanh u - i \tanh^2 u + i + \tanh u}{1 + \tanh^2 u}$$

$$= \frac{2 \tanh u + i(1 - \tanh^2 u)}{1 + \tanh^2 u}.$$

Equating real and imaginary quantities,

$$x = \frac{2 \tanh u}{1 + \tanh^2 u} = \tanh 2u.$$

$$y = \frac{1 - \tanh^2 u}{1 + \tanh^2 u} = \frac{\operatorname{sech}^2 u}{1 + \frac{\sinh^2 u}{\cosh^2 u}}$$

$$= \frac{1}{\cosh^2 u + \sinh^2 u}$$

$$= \frac{1}{\cosh 2u} = \operatorname{sech} 2u.$$

From these results

$$x^2 + y^2 = \tanh^2 2u + \operatorname{sech}^2 2u = 1,$$

i.e. on the Argand diagram the point z lies on the circle $x^2 + y^2 = 1$.

42. The nth roots of a complex number.

If any complex number be given in the form $(a + ib)$ it must first be converted into the form $r(\cos\theta + i \sin\theta)$ before proceeding to find its nth roots.

Let the given complex number be $(a + ib)$ of modulus r and argument θ.

Then $a + ib = r(\cos\theta + i\sin\theta)$

$= r[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]$, where k is any integer.

$\therefore\ (a + ib)^{1/n} = r^{1/n}[\cos 2k\pi + \theta) + i\sin(2k\pi + \theta)]^{1/n}$,

where $r^{1/n}$ is taken to be the positive real nth root of r.

Hence, using Demoivre's theorem,

$$(a + ib)^{1/n} = r^{1/n}\left[\cos\frac{2k\pi + \theta}{n} + i\sin\frac{2k\pi + \theta}{n}\right].$$

This result will give all the nth roots of the complex number provided that k is given the values 0, 1, 2, 3, . . . , $(n - 1)$, or any other n consecutive integral values.

All other values of k will give repetitions of these roots, for when $k = n + m$, the root is

$$r^{1/n}\left[\cos\frac{2(n + m)\pi + \theta}{n} + i\sin\frac{2(n + m)\pi + \theta}{n}\right]$$

$$= r^{1/n}\left[\cos\left(2\pi + \frac{2m\pi + \theta}{n}\right) + i\sin\left(2\pi + \frac{2m\pi + \theta}{n}\right)\right]$$

$$= r^{1/n}\left[\cos\frac{2m\pi + \theta}{n} + i\sin\frac{2m\pi + \theta}{n}\right],$$

which is a repetition of the value of the root when $k = m$.

Hence the n separate roots are given by

$$r^{1/n}\left[\cos\frac{2k\pi + \theta}{n} + i\sin\frac{2k\pi + \theta}{n}\right],$$

where $k = 0, 1, 2, \ldots, (n - 1)$.

Example 22.—Find all the values of $(-1)^{1/5}$ and show that their representative points on the Argand diagram lie on a circle.

$$(-1) = 1[\cos\pi + i\sin\pi] = 1[\cos(2k + 1)\pi + i\sin(2k + 1)\pi],$$

where k is any integer.

$$\therefore\ (-1)^{1/5} = 1\left[\cos\frac{2k + 1}{5}\pi + i\sin\frac{2k + 1}{5}\pi\right], \text{ where } k = 0, 1, 2, 3, 4.$$

The root when $k = 0$ is $\cos\pi/5 + i\sin\pi/5 = 0{\cdot}80902 + i\,0{\cdot}58779$.

The root when $k = 1$ is $\cos 3\pi/5 + i\sin 3\pi/5 = -0{\cdot}30902 + i\,0{\cdot}95106$.

The root when $k = 2$ is $\cos\pi + i\sin\pi = -1$.

The root when $k = 3$ is $\cos 7\pi/5 + i\sin 7\pi/5 = -0{\cdot}30902 - i\,0{\cdot}95106$.

The root when $k = 4$ is $\cos 9\pi/5 + i\sin 9\pi/5 = +0{\cdot}80902 - i\,0{\cdot}58779$.

7. If p and q are any two positive integers, show that

$$\log_e p - \log_e q = 2\left[\frac{p-q}{p+q} + \frac{1}{3}\left(\frac{p-q}{p+q}\right)^3 + \ldots\right].$$

If, when this formula is used to calculate the value of $\log_e 2$, the first three terms only of the series be used, show that the error in the result is less than 0·00015. Thus find $\log_e 2$ and $\log_e 10$, each to three places of decimals.

8. Expand $e^{(1+x)\log_e(1+x)}$, where $|x| < 1$, in ascending powers of x as far as the term in x^4. If x be small enough for x^5 and higher powers of x to be neglected, show that, to this approximation,

$$(1+x)^{1+x} = 1 + x - x\log_e(1-x).$$

9. By use of the expansion for $\log_e(1+y)$, show that, when x is large,

$$\left(1+\frac{1}{x}\right)^{x+1/2} = e^{1+1/(12x^2)} \text{ approximately.}$$

10. Write down the series in ascending powers of x for (i) $\log_e(1+x)$, (ii) $\sin x$, and state the range of values of x for which the series is convergent in each case.

Show that $\quad \log_e \dfrac{\sin x}{x} = -\left(\dfrac{x^2}{6} + \dfrac{x^4}{180} + \dfrac{x^6}{2835} + \ldots\right),$

and hence, or otherwise, find the value of

$$\operatorname*{Lt}_{x\to 0} \frac{\log_e \dfrac{\sin x}{x} + x^2/6}{x(2\sin x - \sin 2x)}.$$

11. Obtain the series in powers of x and y respectively for (i) $\sin x$, (ii) $\log_e(1+y)$, and state for what values of the variable each is valid. Deduce that, when x is small,

$$\log(\sin x) = \log x - \frac{x^2}{6} - \frac{x^4}{180}.$$

12. Write down the series for $\sin\theta$ in ascending power of θ, and prove that it is convergent for all values of θ.

An arc of a circle of unit radius subtends an angle θ at the centre. If a be the chord of the arc and b be the chord of half the arc, prove that, if powers of θ above the fifth be neglected,

$$\theta^2 = (16b - 5a)a/3.$$

If θ be taken as $\pi/4$, show that this formula gives a value of π^2 which is too low, and find the percentage error.

13. If a, b, c are real, different and positive, prove that there are two different real values of x which satisfy the equation $a\cosh x + b\sinh x = c$, only if $b^2 < a^2 < b^2 + c^2$.

If α and β are these two values of x, prove that

$$\cosh\frac{\alpha+\beta}{2} = \frac{a}{\sqrt{(a^2-b^2)}}, \quad \cosh\frac{\alpha-\beta}{2} = \frac{c}{\sqrt{(a^2-b^2)}}.$$

14. Define the hyperbolic sine, cosine and tangent of the angle x, and show that

$$\cosh^{-1}x = \pm \log_e \{x + \sqrt{(x^2 - 1)}\},\ \tanh^{-1} x = \tfrac{1}{2} \log_e \frac{1+x}{1-x}.$$

and deduce that $\tanh^{-1} (\sin \theta) = \cosh^{-1} (\sec \theta)$.

Find, without using tables, the value of $\sinh^{-1} \frac{3}{4}$ to five significant figures.

15. If $t = \tanh \frac{1}{2}x$, prove that $\sinh x = 2t/(1 - t^2)$, $\cosh x = (1 + t^2)/(1 - t^2)$. Find $\tanh x$ if $5 \sinh x - \cosh x = 5$.

16. Define $\sinh x$, $\cosh x$, and $\tanh x$. From your definitions show that, if $t = \tanh x/2$, then $\sinh x = 2t/(1 - t^2)$, and $\cosh x = (1 + t^2)/(1 - t^2)$.

Find all the values of x for which $7 \sinh x + 20 \cosh x = 24$.

17. Define $\sinh x$ and $\cosh x$.

Sketch a graph of $\sinh x$ and show that it is possible to find an angle θ between $-\pi/2$ and $+\pi/2$, for every value of x, such that $\tan \theta = \sinh x$.

Express $\cosh x$ and $\tanh x$ in terms of θ, and show that

$$x = \log_e\left\{\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right\}.$$

18. If u be positive and θ an acute angle such that $\cosh u = \sec \theta$, find $\sinh u$ in terms of θ, and $\sin \theta$ in terms of u.

Prove that (a) $u = \log_e (\sec \theta + \tan \theta)$,

(b) $\theta = \dfrac{\pi}{2} - 2 \tan^{-1}(e^{-u})$,

(c) $\int \sec \theta \, d\theta = [u]$,

and (d) $\int \operatorname{sech} u \, du = [\theta]$.

19. Define the modulus and argument of a complex number.

Prove that

(i) if $| z_1 + z_2 | = |z_1 - z_2 |$, the difference of the amplitudes of z_1 and z_2 is $\pi/2$;

(ii) if $\operatorname{amp} \left\{\dfrac{z_1 + z_2}{z_1 - z_2}\right\} = \frac{1}{2}\pi$, then $| z_1 | = |z_2 |$.

P is the point on the Argand diagram representing the complex number z and Q is the point representing $\dfrac{1}{z-3} + \dfrac{17}{3}$. Find the locus of Q as P describes the circle $| z - 3 | = 3$.

20. If $z = 1 + i$, where $i = \sqrt{(-1)}$, find z^2, z^3, and $1/z$, and plot these values on the Argand diagram. Find also the cube roots of i, and plot them on the Argand diagram.

21. Express in the form $a + ib$, where $i^2 = -1$,

$$\text{(i)}\ \frac{(1 + i)(2 + i)}{(3 + i)}, \quad \text{(ii)}\ \sqrt{\frac{1 + i}{1 - i}}, \quad \text{(iii)}\ \cos\left(\frac{\pi}{4} + \frac{i}{2}\right).$$

If $x + iy = t + 1/t$, and $t = re^{-i\theta}$, show that the locus in the xy plane, corresponding to $r =$ constant, is an ellipse, stating the lengths of its semi-axes, and determine the locus corresponding to $\theta = \pi/4$, when r varies.

22. If $u + iv = a/z$, where $z = x + iy$, $i^2 = -1$, and a is real, show that the curves in the xy plane, along which u and v are respectively constants, are circles, and that they intersect orthogonally.

23. Prove that the amplitude (or argument) of the product of two complex numbers is the sum of their amplitudes (or arguments), and that the amplitude (or argument) of their quotient is the difference of their amplitudes (or arguments).

If $(z - a)/(z + a)$ is pure imaginary, where z is complex and a is a real constant, prove that in the Argand diagram z lies on the circle on $+a$, $-a$ as diameter.

24. (i) If $Z = 1 + 2i$, and $z = 2 + i$, where $i^2 = -1$, exhibit on an Argand diagram Z, z, $p = Zz$, and $q = Z/z$.

(ii) If $x + iy = u + a^2/u$ and $u = a(3 + e^{i\theta})/2$, a and θ being real, find the explicit expressions for x and y in terms of a and θ.

25. If $\left(\dfrac{z+c}{z-c}\right)^2 = \dfrac{z_1 + 2c}{z_1 - 2c}$, where $z = x + iy$, and x, y, c are real, prove that, when $z = ce^{i\theta}$, then $z_1 = 2c \cos \theta$.

Hence, or otherwise, prove that, if the point representing z describes the circle $x^2 + y^2 = c^2$, the point representing z_1 describes twice the segment of the x-axis from $2c$ to $-2c$, once in either direction.

26. Define the modulus and argument (amplitude) of a complex number.

Two complex numbers z_1 and z_2 are represented by points on an Argand diagram. Show how to construct geometrically the points which represent $(z_1 + z_2)$ and $(z_1 - z_2)$.

Interpret geometrically, or otherwise, the following loci on the Argand diagram:

(i) $|z + 3i|^2 - |z - 3i|^2 = 12$,

(ii) $|z + ik|^2 + |z - ik|^2 = 10k^2 \ (k > 0)$,

(iii) $\operatorname{amp} \left\{\dfrac{z-2}{z+2}\right\} = \frac{1}{3}\pi$,

where $z = x + iy$ in each case.

27. Explain the method of representing a complex number by a point in the Argand diagram, and define the meaning of the terms modulus and amplitude (argument).

If a be a complex number, and r and θ be real, show that the point representing z, where r is constant and $z = a + re^{i\theta}$, lies on a fixed circle, whose centre is a, for all values of θ.

Let T be the length of the tangent from the point representing Z to this circle. If $Z = a + Re^{i\phi}$, where R and ϕ are real and $R > r$, show that $\sqrt{T^2 + r^2} = \operatorname{mod}(Z - a)$. Explain why the last result is independent of ϕ.

28. (i) Explain how a complex number may be represented on an Argand diagram, and show how the sum and difference of two given complex numbers may be represented.

Prove that, if z and a are two complex numbers,

$$|z + a|^2 + |z - a|^2 = 2\{|z|^2 + |a|^2\}.$$

(ii) The complex number z is represented by a point on the circle whose centre is at the point $1 + 0 . i$, and whose radius is unity. Show in a diagram how the number $z - 2$ may be represented, and prove that $(z - 2)/z = i \tan(\operatorname{amp} z)$.

29. Show how to obtain on the Argand diagram by a geometrical construction (i) the product, (ii) the quotient of two given complex numbers.

Prove that, if the ratio $(z - i)/(z - 1)$ is purely imaginary, then the point z lies on a circle whose centre is the point $\frac{1}{2}(1 + i)$ and whose radius is $\frac{1}{2}\sqrt{2}$.

30. If P be a point in an Argand diagram representing the complex quantity z, and Q be the point representing $\dfrac{1}{z} + \dfrac{7}{4} - i$, find the locus of Q as P describes the circle with centre $(2, 0)$ and radius 2.

31. If z, z_1, z_2 are three complex quantities, where z_1 and z_2 are constant, and z varies so that the amplitude of $(z - z_1)/(z - z_2)$ is constant, show that, in the Argand diagram, the point representing z describes an arc of a circle.

If $z_1 = 2(3 + i)$, $z_2 = 2(1 + i)$, and the amplitude of $\dfrac{z - z_1}{z - z_2}$ is $\dfrac{\pi}{6}$, show that the modulus of $z - 2\{2 + i(1 + \sqrt{3})\}$ is constant, and find its value.

32. If the point P on the Argand diagram represents the complex quantity z, show how to construct geometrically the point Q representing z^2.

If P lie on the circle, centre $(1, 0)$ passing through $(0, 0)$, show that $|z^2 - z| = |z|$ and that $\arg(z - 1) = \arg z^2 = \frac{2}{3}\arg(z^2 - z)$.

33. In an Argand diagram the points P and Q represent the complex numbers z_1 and z respectively, where $z_1 = \dfrac{z - 1}{z + 1}$.

Find the locus of Q if P describes a line through O inclined at an angle α to the x-axis, and show that, if Q describes a circle of a coaxal system whose limiting points are $(1, 0)$, $(-1, 0)$, then P describes a circle whose centre is at the origin.

34. Prove $(\cos\alpha_1 + i\sin\alpha_1)(\cos\alpha_2 + i\sin\alpha_2)\ldots(\cos\alpha_n + i\sin\alpha_n)$
$$= \cos(\alpha_1 + \alpha_2 + \ldots + \alpha_n) + i\sin(\alpha_1 + \alpha_2 + \ldots + \alpha_n).$$

Factorize $bc(b - c) + ca(c - a) + ab(a - b)$ and, by means of the substitutions $a = \cos 2\alpha + i\sin 2\alpha$, $b = \cos 2\beta + i\sin 2\beta$, $c = \cos 2\gamma + i\sin 2\gamma$, prove that $\cos(\beta + \gamma - 2\alpha)\sin(\beta - \gamma) + \cos(\gamma + \alpha - 2\beta)\sin(\gamma - \alpha) + \cos(\alpha + \beta - 2\gamma) \times \sin(\alpha - \beta) = 4\sin(\beta - \gamma)\sin(\gamma - \alpha)\sin(\alpha - \beta)$.

35. State Demoivre's theorem and prove it for (i) a positive integral exponent, (ii) a fractional exponent.

Express $\sin 9\theta/\sin\theta$ as a polynomial in $\cos\theta$ and deduce, or prove otherwise, that

$$\text{(i)}\quad \sec^2\frac{\pi}{9} + \sec^2\frac{2\pi}{9} + \sec^2\frac{4\pi}{9} = 36,$$

$$\text{(ii)}\quad \sec\frac{\pi}{9}\sec\frac{2\pi}{9}\sec\frac{4\pi}{9} = 8.$$

36. By use of Demoivre's theorem, or otherwise, expand $\cos 8\theta$ in ascending powers of $\sin^2\theta$.

Show that $\operatorname{cosec}^2\dfrac{\pi}{16} + \operatorname{cosec}^2\dfrac{3\pi}{16} + \operatorname{cosec}^2\dfrac{5\pi}{16} + \operatorname{cosec}^2\dfrac{7\pi}{16} = 32.$

37. If $\tanh(u + iv) = x + iy$, where u, v, x, y are real, find the values of x and y in terms of u and v.

If $x = y = 1$, find the values of u and v.

38. If $\tan\frac{1}{2}(x + iy) = u + iv$, where x, y, u, v are real, find the values of u and v in terms of x and y.

If $-\pi/2 < x < \pi/2$, prove that $u^2 + v^2 < 1$.

39. If $x + iy = \cos(u + iv)$, where x, y, u, v are real, prove that

$$(1 + x)^2 + y^2 = (\cosh v + \cos u)^2;\ (1 - x)^2 + y^2 = (\cosh v - \cos u)^2.$$

If in the above $x = \cos\theta$, $y = \sin\theta$, where $0 < \theta < \pi$, find the values of $\cos u$ and $\cosh v$ in terms of $\cos\theta/2$ and $\sin\theta/2$, justifying the choice of sign when square roots are taken.

40. Express $e^{x\cos\theta}\sin(x\sin\theta)$, where x and θ are real, in the form

$$\frac{1}{2i}\left[e^{(p+iq)x} - e^{(p-iq)x}\right],$$

and hence obtain the coefficient of x in the expansion of the function in ascending powers of x.

41. Define $\sinh z$ and $\cosh z$ and obtain the relations between them and the functions $\sin iz$, $\cos iz$.

In an Argand diagram the point P represents the complex quantity $2\cosh\left(\lambda + \frac{i\pi}{4}\right)$, where λ is a real variable. Find the Cartesian equation of the locus of P, and the values of $2\cosh\left(\lambda + \frac{i\pi}{4}\right)$ when its modulus is four units.

42. If $\tan^{-1}(x + iy) = u + iv$, where x, y, u, v are all real, show that

$$u = \tfrac{1}{2}\tan^{-1}\frac{2x}{1 - x^2 - y^2},\quad v = \tfrac{1}{2}\tanh^{-1}\frac{2y}{1 + x^2 + y^2}.$$

Show that, if the real part of $\tan^{-1}(x + iy)$ is $\pi/8$, the representative point of the complex number on an Argand diagram lies on a circle of radius $\sqrt{2}$, with its centre at the point $-1 + 0 . i$.

43. If $x + iy = \cos(u + iv)$, express x and y in terms of u and v.

Show that $\cos^2 u$ and $\cosh^2 v$ are the roots of the equation

$$\lambda^2 - (x^2 + y^2 + 1)\lambda + x^2 = 0.$$

If $\cos(u + iv) = (5 + 4i\sqrt{3})/6$, find u and v.

44. Express $\sin ix$, $\cos ix$ in terms of $\sinh x$, $\cosh x$ respectively.

If $x + iy = \tan(u + iv)$, prove that

$$\text{(i)}\ \coth 2v = \frac{x^2 + y^2 + 1}{2y},\quad \text{(ii)}\ \cot 2u = \frac{1 - x^2 - y^2}{2x}.$$

*45. (i) If $x + iy = \cosh(u + iv)$, express x and y in terms of u and v and find the equations of the curves $u =$ constant, $v =$ constant.

(ii) By means of the substitution $\phi = \frac{1}{2}\pi - 2t$, or otherwise, prove that

$$\left(\frac{1 + \sin\phi + i\cos\phi}{1 + \sin\phi - i\cos\phi}\right)^n = \cos(\tfrac{1}{2}n\pi - n\phi) + i\sin(\tfrac{1}{2}n\pi - n\phi).$$

46. Express $\log_e(x + iy)$ in the form $(a + ib)$.

If $u + iv = \log_e\{(z - a)/(z + a)\}$, where $z = x + iy$ and a is a real constant, find the values of u and v in terms of x and y, and show that $u =$ constant, and $v =$ constant, represent in the xy plane two sets of orthogonal circles.

47. Express $x^{2n}+1$ as a product of n real quadratic factors.

By putting $x=e^{i\theta}$ in this expression, prove that

$$\cos n\theta = 2^{n-1}\prod_{k=0}^{k=n-1}\left(\cos\theta-\cos\frac{2k+1}{2n}\pi\right).$$

48. Express $x^{2n}+1$ as a product of n real quadratic factors. Prove that the roots of $(1+x)^{2n}+(1-x)^{2n}=0$ are all pure imaginaries.

49. (i) Solve the equation $(x+1)^8+x^8=0$.

(ii) Prove that, if n be a positive integer,

$$x^{2n}-1=(x^2-1)\prod_{s=1}^{s=n-1}\left\{x^2-2x\cos\frac{s\pi}{n}+1\right\},$$

and deduce, or prove otherwise, that

$$\frac{\sinh n\theta}{\sinh\theta}=2^{n-1}\prod_{s=1}^{s=n-1}\left(\cosh\theta-\cos\frac{s\pi}{n}\right).$$

50. Prove that $x^{2n}-2x^n\cos n\theta+1=\displaystyle\prod_{r=0}^{r=n-1}\left\{x^2-2x\cos\left(\theta+\frac{2r\pi}{n}\right)+1\right\}.$

Deduce, or prove otherwise, that

$$\text{(i)}\quad \cos n\theta-\cos n\alpha=2^{n-1}\prod_{r=0}^{r=n-1}\left\{\cos\theta-\cos\left(\alpha+\frac{2r\pi}{n}\right)\right\};$$

$$\text{(ii)}\quad 1-\sin^2\frac{n\theta}{2}\operatorname{cosec}^2\frac{n\alpha}{2}=\prod_{r=0}^{r=n-1}\left\{1-\sin^2\frac{\theta}{2}\operatorname{cosec}^2\left(\frac{\alpha}{2}+\frac{r\pi}{n}\right)\right\}.$$

CHAPTER III

Partial Fractions, and Summation of Series

Partial Fractions

1. Consider $\dfrac{2}{x-3}+\dfrac{1}{2x+1}$, the result of which is $\dfrac{5x-1}{(x-3)(2x+1)}$.

It can be said that $\dfrac{5x-1}{(x-3)(2x+1)}=\dfrac{2}{x-3}+\dfrac{1}{2x+1}$, i.e. the fraction $\dfrac{5x-1}{(x-3)(2x+1)}$ can be expressed as the sum of two simpler fractions, and in this case the fraction $(5x-1)/\{(x-3)(2x+1)\}$ is said to be expressed in *partial fractions* which are $\dfrac{2}{x-3}$ and $\dfrac{1}{2x+1}$. In general, if $f(x)$ and $\phi(x)$ be two rational, integral, algebraical functions of x (i.e. a polynomial in x of the form $a_0+a_1x+a_2x^2+\ldots+a_nx^n$, where the coefficients are rational) and the fraction $f(x)/\phi(x)$ be expressed as the algebraic sum of simpler fractions according to *certain specified rules* which are given later, the fraction $f(x)/\phi(x)$ is then said to be resolved into partial fractions.

2. Rules to be observed.

There are two cases to be considered:

(i) degree of $f(x)<$ degree of $\phi(x)$,

(ii) degree of $f(x)\geq$ degree of $\phi(x)$,

and in all that follows the A's, B's, etc., are constants.

Case (i).

(*a*) To every linear factor (a_1x+b_1) of $\phi(x)$ there will be a corresponding partial fraction $A_1/(a_1x+b_1)$.

(*b*) To every quadratic factor $a_2x^2+b_2x+c_2$ of $\phi(x)$ there will be a corresponding partial fraction $\dfrac{A_2x+B_2}{a_2x^2+b_2x+c_2}$.

(*c*) To every repeated linear factor $(a_3x + b_3)^2$ of $\phi(x)$ there will be two corresponding partial fractions $\dfrac{A_3}{a_3x + b_3} + \dfrac{B_3}{(a_3x + b_3)^2}$.

(*d*) To every repeated quadratic factor $(a_4x^2 + b_4x + c_4)^2$ of $\phi(x)$ there will be two corresponding partial fractions

$$\frac{A_4x + B_4}{a_4x^2 + b_4x + c_4} + \frac{C_4x + D_4}{(a_4x^2 + b_4x + c_4)^2}.$$

(*e*) To every thrice-repeated linear factor $(a_5x + b_5)^3$ of $\phi(x)$ there will be three corresponding partial fractions

$$\frac{A_5}{a_5x + b_5} + \frac{B_5}{(a_5x + b_5)^2} + \frac{C_5}{(a_5x + b_5)^3}.$$

(*f*) To every cubic factor $a_6x^3 + b_6x^2 + c_6x + d_6$ of $\phi(x)$ there will be a corresponding partial fraction

$$\frac{A_6x^2 + B_6x + C_6}{a_6x^3 + b_6x^2 + c_6x + d_6},$$

and so on.

N.B.—A quadratic factor or cubic factor cannot be factorized further into *rational* factors.

Case (ii).

(*a*) If $f(x)$ be of the same degree as $\phi(x)$, then A will be added to the partial fractions as given by Case (i).

(*b*) If $f(x)$ be one degree higher than $\phi(x)$, then $(Ax + B)$ will be added to the partial fractions as given by Case (i).

(*c*) If $f(x)$ be two degrees higher than $\phi(x)$, then $(Ax^2 + Bx + C)$ will be added to the partial fractions as given by Case (i); and so on.

Note.—The rules are specially chosen to fit in with differentiation and integration, and should be strictly adhered to.

N.B.—The constants (A's, B's, etc.) are determined in the following manner:

Let the given fraction $f(x)/\phi(x)$ be identically equated to the sum of its corresponding partial fractions as given under Cases (i) and (ii), and then multiply through this identity by $\phi(x)$, thus obtaining a second identity. The values of the unknown constants can be then determined from this second identity by adopting one of the following methods:

(*a*) give suitable values to x;

(*b*) equate corresponding coefficients (starting with the highest-powered terms);

(*c*) combine methods (*a*) and (*b*).

When there are linear factors of $\phi(x)$, it greatly facilitates the working if x be given the values (in succession) that make each of these linear factors vanish. Another useful value to give to x is zero.

Example 1.—Express in partial fractions

$$(a)\quad \frac{4x+21}{(x-1)(x+4)},\qquad (b)\quad \frac{5x^2-3x+4}{(x+1)(x^2-2x+6)},$$

$$(c)\quad \frac{x^2+x+1}{x^2+2x+1},\qquad (d)\quad \frac{x^4+2x+4}{(2x^2+3)(x-2)},$$

and obtain the expansion of (*c*) and (*d*) in positive integral powers of x as far as the term in x^4.

(*a*) Let $$\frac{4x+21}{(x-1)(x+4)} \equiv \frac{A}{x-1} + \frac{B}{x+4}.$$

Then $$4x+21 \equiv A(x+4) + B(x-1).$$

Since this is an identity, using

$$x = 1,\quad 25 = 5A,\quad \therefore A = 5;$$
$$x = -4,\quad 5 = -5B,\quad \therefore B = -1.$$
$$\therefore \quad \frac{4x+21}{(x-1)(x+4)} \equiv \frac{5}{(x-1)} - \frac{1}{(x+4)}.$$

(*b*) Let $$\frac{5x^2-3x+4}{(x+1)(x^2-2x+6)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2-2x+6}.$$

Then $$5x^2-3x+4 \equiv A(x^2-2x+6) + (x+1)(Bx+C).$$

Since this is an identity, using

$$x = -1,\quad 12 = 9A,\quad \therefore A = 4/3;$$
$$x = 0,\quad 4 = 6A + C = 8 + C,\quad \therefore C = -4.$$

Equating coefficients of x^2, $5 = A + B = \frac{4}{3} + B$, $\therefore B = 11/3$.

Hence $$\frac{5x^2-3x+4}{(x+1)(x^2-2x+6)} \equiv \frac{4}{3(x+1)} + \frac{11x-12}{3(x^2-2x+6)}.$$

(*c*) Let $$\frac{x^2+x+1}{x^2+2x+1} \equiv A + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Then $$x^2+x+1 \equiv A(x+1)^2 + B(x+1) + C.$$

Since this is an identity, using

$$x = -1,\quad 1 = C,\quad \therefore C = 1.$$

Equating coefficients of x^2, $1 = A$, $\therefore A = 1$.

" " x, $1 = 2A + B = 2 + B$, $\therefore B = -1$.

$$\therefore \frac{x^2 + x + 1}{x^2 + 2x + 1} \equiv 1 - \frac{1}{x+1} + \frac{1}{(x+1)^2}.$$

$$\begin{aligned}\frac{x^2 + x + 1}{x^2 + 2x + 1} &\equiv 1 - (1 + x)^{-1} + (1 + x)^{-2}\\ &\equiv 1 - \{1 - x + x^2 - x^3 + x^4 - \ldots\}\\ &\qquad + \{1 - 2x + 3x^2 - 4x^3 + 5x^4 - \ldots\}\\ &\equiv 1 - x + 2x^2 - 3x^3 + 4x^4 \text{ as far as term in } x^4.\end{aligned}$$

(*d*) Let $$\frac{x^4 + 2x + 4}{(2x^2 + 3)(x - 2)} \equiv Ax + B + \frac{C}{x - 2} + \frac{Dx + E}{2x^2 + 3}.$$

Then $x^4 + 2x + 4 \equiv (Ax + B)(2x^2 + 3)(x - 2) + C(2x^2 + 3) + (Dx + E)(x - 2)$.

From this identity, using

$$x = 2, \quad 24 = 11C, \quad \therefore C = 24/11.$$

Equating coefficients of x^4, $1 = 2A$, $\therefore A = 1/2$.

" " x^3, $0 = -4A + 2B = -2 + 2B$, $\therefore B = +1$.

" " x^2, $0 = 3A - 4B + 2C + D$

$= \frac{3}{2} - 4 + \frac{48}{11} + D$, $\therefore D = -\frac{41}{22}$.

" " x, $2 = -6A + 3B - 2D + E$

$= -3 + 3 + \frac{41}{11} + E$, $\therefore E = -\frac{19}{11}$.

Check.—Coefficient of unity L.H.S. $= 4$;

$$\text{R.H.S.} = -6B + 3C - 2E = -6 + \tfrac{72}{11} + \tfrac{38}{11} = 4.$$

$$\therefore \frac{x^4 + 2x + 4}{(2x^2 + 3)(x - 2)} \equiv \frac{x}{2} + 1 + \frac{24}{11(x - 2)} - \frac{(41x + 38)}{22(2x^2 + 3)}.$$

Using this result,

$$\begin{aligned}\frac{x^4 + 2x + 4}{(2x^2 + 3)(x-2)} &= 1 + \frac{x}{2} - \frac{24}{22}(1 - x/2)^{-1} - \frac{(41x + 38)}{66}\left(1 + \frac{2}{3}x^2\right)^{-1}\\ &= 1 + \frac{x}{2} - \frac{12}{11}\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{16}\right)\\ &\qquad - \frac{41x + 38}{66}\left(1 - \frac{2}{3}x^2 + \frac{4}{9}x^4 - \ldots\right)\\ &= 1 + \frac{x}{2} - \frac{12}{11} - \frac{6}{11}x - \frac{3}{11}x^2 - \frac{3}{22}x^3 - \frac{3}{44}x^4\\ &\qquad - \tfrac{41}{66}x + \tfrac{41}{99}x^3 - \tfrac{19}{33} + \tfrac{38}{99}x^2 - \tfrac{76}{297}x^4 + \ldots\\ &= -\tfrac{2}{3} - \tfrac{2}{3}x + \tfrac{1}{9}x^2 + \tfrac{5}{18}x^3 - \tfrac{35}{108}x^4\end{aligned}$$

as far as the term in x^4.

Example 2 (L.U.).—Express $\dfrac{27}{(1-5x+4x^2)^2}$ in terms of partial fractions and obtain its expansion in positive integral powers of x, stating the range of values of x for which the expansion is valid.

Deduce that $4^{n+2}(3n+1)+3n+11$ has a factor 27 for all positive integral values of n.

$$\frac{27}{(1-5x+4x^2)^2}=\frac{27}{(1-x)^2(1-4x)^2}.$$

Let $$\frac{27}{(1-5x+4x^2)^2}\equiv\frac{A}{1-x}+\frac{B}{(1-x)^2}+\frac{C}{(1-4x)}+\frac{D}{(1-4x)^2},$$

then $27\equiv A(1-x)(1-4x)^2+B(1-4x)^2+C(1-x)^2(1-4x)+D(1-x)^2.$

In this identity, using

$$x=1,\quad 27=9B,\quad \therefore B=3;$$
$$x=\tfrac{1}{4},\quad \tfrac{9}{16}D=27,\quad \therefore D=48.$$

Equating coefficients of x^3, $0=-16A-4C$, $\quad\therefore 4A+C=0.\ \ .\ \ .\ \ .\quad$ (i)

" " x^2, $0=24A+16B+9C+D=24A+48+9C+48,$

$\therefore\ 24A+9C+96=0,$

i.e. $8A+3C+32=0.\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\quad$ (ii)

(ii) $-$ (i) $\times$ 3 gives $\quad -4A+32=0,\quad\therefore A=8.$

From (i) $\quad C=-32.$

$$\therefore\ \frac{27}{(1-5x+4x^2)^2}=\frac{8}{1-x}+\frac{3}{(1-x)^2}-\frac{32}{(1-4x)}+\frac{48}{(1-4x)^2}.$$

Using this result and the binomial expansion

$$\frac{27}{(1-5x+4x^2)^2}\equiv 8(1-x)^{-1}+3(1-x)^{-2}-32(1-4x)^{-1}+48(1-4x)^{-2}$$
$$=8(1+x+x^2+x^3+\ldots+x^n+\ldots)$$
$$+3(1+2x+3x^2+4x^3+\ldots+(n+1)x^n+\ldots)$$
$$-32(1+4x+16x^2+64x^3+\ldots+4^nx^n+\ldots)$$
$$+48(1+2\,.\,4x+3\,.\,16x^2+4\,.\,64x^3+\ldots+(n+1)4^nx^n+\ldots)$$
$$=27+270x+1809x^2+\ldots+\{8+3n+3+4^n(48n+16)\}x^n+\ldots$$
$$=27\{1+10x+67x^2+\ldots\}.$$

Now $(1-x)^{-1}$ and $(1-x)^{-2}$ are valid for $|x|<1$, and $(1-4x)^{-1}$ and $(1-4x)^{-2}$ are valid for $|4x|<1$, i.e. $|x|<\tfrac{1}{4}$. Therefore the series is valid for the combined range $|x|<\tfrac{1}{4}$.

Now $27/(1-5x+4x^2)^2$ can be expanded as $27\{1-5x+4x^2\}^{-2}$, i.e. as $27\{1-x\}^{-2}\{1-4x\}^{-2}$, and in the expansion of $(1-x)^{-2}(1-4x)^{-2}$ by the binomial theorem it is clear that the coefficient of x^n will be a positive integer; therefore in the expansion of $27/(1-5x+4x^2)^2$ in ascending powers, the coefficient of x^n will be 27 $\times$ some integer, i.e. 27 will be a factor of the coefficient of x^n.

From the previous working the coefficient of x^n is

$$8+3n+3+4^n(48n+16)=11+3n+4^{n+2}(3n+1).$$

$\therefore\ 11+3n+4^{n+2}(3n+1)$ *has a factor* 27 for all positive integral values of n.

SUMMATION OF SERIES

The methods for the summation of a series vary according to the series and are classified as follows:

A. Inspection Method (order of testing)

3. (i) Test for an A.P.

(ii) Test for a G.P.

(iii) Test for an arithmetico-geometrical progression.

(iv) Test for a binomial expansion.

(v) Test for an exponential series (this includes sine, cosine, sinh, and cosh series).

(vi) Test for a logarithmic series.

Note.—The general arithmetico-geometrical progression is a series of the form

$$a + (a + d)x + (a + 2d)x^2 + (a + 3d)x^3 + \dots + (a + \overline{n - 1}d)x^n + \dots,$$

where the coefficients of the powers of x are in A.P., and the x portions of the terms are in G.P.

The method of summation of this series is the same as for the G.P., namely:

$$\text{Let } S = a + (a + d)x + (a + 2d)x^2 + \dots + (a + \overline{n - 2}d)x^{n-2} + (a + \overline{n - 1}d)x^{n-1} \quad . \quad . \quad . \quad (1)$$

$$\therefore\ xS = ax + (a + d)x^2 + \dots + (a + \overline{n - 2}d)x^{n-1} + (a + \overline{n - 1}d)x^n. \quad . \quad . \quad . \quad (2)$$

(1) — (2) gives

$$S(1 - x) = a + dx + dx^2 + dx^3 + \dots + dx^{n-1} - (a + \overline{n - 1}d)x^n$$

$$= a - (a + \overline{n - 1}d)x^n + dx(1 + x + x^2 + \dots + x^{n-2})$$

$$= a - (a + \overline{n - 1}d)x^n + dx\frac{(1 - x^{n-1})}{1 - x},$$

$$\therefore\ S = \frac{a - (a + \overline{n - 1}d)x^n}{(1 - x)} + dx\frac{(1 - x^{n-1})}{(1 - x)^2}.$$

In the case of the sum of a series which can be seen to be a binomial expansion, assume that the sum of the series is $(a + x)^n$, whose expansion is $a^n + na^{n-1}x + \frac{n(n - 1)}{2!}a^{n-2}x^2 + \dots$.

By comparing this series term for term with the given series (first three terms sufficient), the values of x, a, and n can be found.

Example 3.—Find the sum of the binomial series

$$1+\frac{2}{6}+\frac{2.5}{6.12}+\frac{2.5.8}{6.12.18}+\ldots\to\infty.$$

Method (*a*) (setting up method).—Put series in form

$$a^n+na^{n-1}x+\frac{n(n-1)}{2!}x^2+\ldots \text{ by inspection.}$$

$$1+\frac{2}{6}+\frac{2.5}{6.12}+\frac{2.5.8}{6.12.18}+\ldots\to\infty$$

$$=1+(-\tfrac{2}{3})(-\tfrac{1}{2})+\frac{(-\frac{2}{3})(-\frac{5}{3})}{2!}(-\tfrac{1}{2})^2+\frac{(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{3!}(-\tfrac{1}{2})^3+\ldots\to\infty$$

$$=(1-\tfrac{1}{2})^{-2/3}=(\tfrac{1}{2})^{-2/3}=2^{2/3}=\sqrt[3]{4}.$$

Method (*b*).—Since the series is a binomial expansion commencing with unity, let its sum $=(1+x)^n$.

Now $$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots.$$

Comparing the second and third terms in this and the given series

$$nx=\tfrac{1}{3}, \quad \ldots \quad \text{(i)}$$

$$\frac{n(n-1)}{2!}x^2=\frac{5}{36}. \quad \ldots \quad \text{(ii)}$$

(ii) $\div$ (i) gives $$\frac{n-1}{2}x=\frac{5}{12}, \quad \therefore\ nx-x=\frac{5}{6}. \quad \ldots \quad \text{(iii)}$$

From (i) and (iii), $x=-\frac{1}{2}$, and using (i), $n=-\frac{2}{3}$.

$$\therefore \text{ required sum }=(1-\tfrac{1}{2})^{-2/3}=\sqrt[3]{4}.$$

4. For the purposes of revision the exponential and logarithmic series are restated.

Exponential Series.

$$1+x+x^2/2!+x^3/3!+\ldots\to\infty=e^x.$$
$$1-x+x^2/2!-x^3/3!+\ldots\to\infty=e^{-x}.$$
$$1+x^2/2!+x^4/4!+x^6/6!+\ldots\to\infty=\cosh x.$$
$$x+x^3/3!+x^5/5!+x^7/7!+\ldots\to\infty=\sinh x.$$
$$1-x^2/2!+x^4/4!-x^6/6!+\ldots\to\infty=\cos x.$$
$$x-x^3/3!+x^5/5!-x^7/7!+\ldots\to\infty=\sin x.$$
$$1+x\log_e a+x^2(\log_e a)^2/2!+\ldots\to\infty=a^x.$$

These are valid for *all* values of x.

Logarithmic Series.

$$x - x^2/2 + x^3/3 - x^4/4 + \ldots \to \infty = \log_e(1+x), \text{ for } -1 < x \leqslant 1.$$

$$-(x + x^2/2 + x^3/3 + x^4/4 + \ldots \to \infty) = \log_e(1-x),$$
$$\text{for } -1 \leqslant x < 1.$$

$$2(x + x^3/3 + x^5/5 + x^7/7 + \ldots \to \infty) = \log_e \frac{1+x}{1-x}, \text{ for } -1 < x < 1.$$

$$2\left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \ldots \to \infty\right) = \log_e \frac{n+1}{n-1}, \text{ for } n > 1.$$

$$2\left(\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots \to \infty\right) = \log_e \frac{n+1}{n},$$
$$\text{for } n > 0.$$

Example 4 (L.U.)—Sum to infinity the series whose rth term is

$$(2r^2 - 3r - 1)/r!.$$

Note the method of solution.

By inspection

$$\frac{2r^2 - 3r - 1}{r!} = \frac{2r(r-1) - r - 1}{r!} = \frac{2}{(r-2)!} - \frac{1}{(r-1)!} - \frac{1}{r!}. \qquad \text{(i)}$$

When $r = 1$, the term $= -2$ (from original) $= 0 - 1 - 1$.

When $r = 2$, the term $= 1/2!$ (from original) $= 2 - 1 - 1/2!$.

From (i), when $r = 3$, the term $= 2 - 1/2! - 1/3!$;

when $r = 4$, the term $= 2/2! - 1/3! - 1/4!$, etc.

$\therefore$ Sum of series = sum of the columns

$$= 2(1 + 1 + 1/2! + \ldots \to \infty)$$
$$- (1 + 1 + 1/2! + \ldots \to \infty)$$
$$- (1 + 1/2! + 1/3! + 1/4! + \ldots \to \infty)$$
$$= 2e - e - (e - 1) = 1.$$

Example 5 (L.U.).—Show that the sum to n terms of the series whose rth term is $r^2 . 2^r$ is $2^{n+1}(n^2 - 2n + 3) - 6$.

N.B.—This is similar to the arithmetico-geometrical series, and is approached in a similar manner.

Let $S = 1^2 . 2 + 2^2 . 2^2 + 3^2 . 2^3 + \ldots + r^2 . 2^r + \ldots + n^2 . 2^n$. . (i)

$\therefore\ 2S = 1^2 . 2^2 + 2^2 . 2^3 + \ldots + (r-1)^2 . 2^r + \ldots + n^2 . 2^{n+1}$. . (ii)

(ii) — (i) gives

$S = -1^2 . 2 - \{3 . 2^2 + 5 . 2^3 + \ldots + (2n-1)2^n\} + n^2 . 2^{n+1}$. . (iii)

$\therefore\ 2S = -1^2 . 2^2 - \{3 . 2^3 + \ldots + (2n-3)2^n\} - (2n-1)2^{n+1} + n^2 2^{n+2}$. (iv)

(iv) − (iii) gives

$$S = -2 + \{3 . 2^2 + 2 . 2^3 + \ldots + 2 . 2^n\} - (2n - 1) . 2^{n+1} + n^2 . 2^{n+1}$$
$$= 2 + \{2^3 + 2^4 + \ldots + 2^{n+1}\} + 2^{n+1}(n^2 - 2n + 1)$$
$$= 2 + 2^3\{1 + 2 + \ldots + 2^{n-2}\} + 2^{n+1}(n^2 - 2n + 1)$$
$$= 2 + 8\left\{\frac{2^{n-1} - 1}{2 - 1}\right\} + 2^{n+1}(n^2 - 2n + 1)$$
$$= 2 + 2 . 2^{n+1} - 8 + 2^{n+1}(n^2 - 2n + 1)$$
$$= 2^{n+1}(n^2 - 2n + 3) - 6.$$

B. Summation of types not obvious by inspection

5. Summation by means of undetermined coefficients.

This method is best exemplified by the two following theorems.

6. Theorem.—Find the sum of the squares of the first n natural numbers (i.e. 1, 2, 3, . . . , n).

Let $1^2 + 2^2 + 3^2 + \ldots + n^2$

$$\equiv A_0 + A_1 n + A_2 n^2 + A_3 n^3 + \ldots \quad . \quad (3)$$

Replacing n by $(n + 1)$,

$1^2 + 2^2 + 3^2 + \ldots + n^2 + (n + 1)^2$

$$\equiv A_0 + A_1(n + 1) + A_2(n + 1)^2 + A_3(n + 1)^3 + \ldots \quad . \quad (4)$$

(4) − (3) gives

$$(n + 1)^2 \equiv A_1 + A_2(2n + 1) + A_3(3n^2 + 3n + 1) + \ldots \quad . \quad (5)$$

Since (5) is an identity, and the highest power of n on the L.H.S. is the square, it follows that A_4, A_5, etc., must be zero.

Equating coefficients on the two sides of the identity (5),

n^2 $\quad 1 = 3A_3, \quad \therefore A_3 = \frac{1}{3}.$

n $\quad 2 = 2A_2 + 3A_3 = 2A_2 + 1, \quad \therefore A_2 = \frac{1}{2}.$

Unity $\quad 1 = A_1 + A_2 + A_3 = A_1 + \frac{5}{6}, \quad \therefore A_1 = \frac{1}{6}.$

$$\therefore 1^2 + 2^2 + 3^2 + \ldots + n^2 \equiv A_0 + \tfrac{1}{6}n + \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3.$$

Using $n = 1$ in this identity,

$$1 = A_0 + \tfrac{1}{6} + \tfrac{1}{2} + \tfrac{1}{3}, \quad \therefore A_0 = 0.$$

$$\therefore 1^2 + 2^2 + 3^2 + \ldots + n^2 = \tfrac{1}{6}n + \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3$$
$$= \frac{n}{6}(1 + 3n + 2n^2)$$
$$= \boldsymbol{n(n + 1)(2n + 1)/6}.$$

The sum of the squares is usually denoted by S_2, the sum of the natural numbers by S_1, and the sum of their cubes by S_3.

7. Theorem.—To find the sum of the cubes of the first n natural numbers.

Let $1^3 + 2^3 + 3^3 + \ldots + n^3$

$$\equiv A_0 + A_1 n + A_2 n^2 + A_3 n^3 + A_4 n^4 + \ldots \quad . \quad (6)$$

Replacing n by $(n + 1)$ in this identity,

$$1^3 + 2^3 + 3^3 + \ldots + n^3 + (n+1)^3 \equiv A_0 + A_1(n+1)$$
$$+ A_2(n+1)^2 + A_3(n+1)^3 + A_4(n+1)^4 + \ldots \quad . \quad (7)$$

(7) — (6) gives

$$(n+1)^3 \equiv A_1 + A_2(2n+1) + A_3(3n^2+3n+1)$$
$$+ A_4(4n^3 + 6n^2 + 4n + 1) + \ldots \quad . \quad (8)$$

Since the highest power of n on the L.H.S. is the cube, it follows that A_5, A_6, etc., must be zero.

Equating coefficients of n in the identity (8),

n^3 $\quad 1 = 4A_4, \quad \therefore\ A_4 = \frac{1}{4}.$

n^2 $\quad 3 = 3A_3 + 6A_4 = 3A_3 + \frac{3}{2}, \quad \therefore\ A_3 = \frac{1}{2}.$

n $\quad 3 = 2A_2 + 3A_3 + 4A_4$

$\quad\quad = 2A_2 + \frac{3}{2} + 1, \quad \therefore\ A_2 = \frac{1}{4}.$

Unity $\quad 1 = A_1 + A_2 + A_3 + A_4$

$\quad\quad = A_1 + \frac{1}{4} + \frac{1}{2} + \frac{1}{4}, \quad \therefore\ A_1 = 0.$

Thus $\quad 1^3 + 2^3 + 3^3 + \ldots + n^3 \equiv A_0 + \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4.$

Using $n = 1$ in this identity,

$$1 = A_0 + \tfrac{1}{4} + \tfrac{1}{2} + \tfrac{1}{4}, \quad \therefore\ A_0 = 0.$$

Hence $\quad S_3 \equiv \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4$

$$\equiv \left\{\frac{n(n+1)}{2}\right\}^2, \quad \text{i.e. } S_3 = (S_1)^2.$$

Note.—Using the results for S_1, S_2, S_3 it is possible to find the sum of various other series.

Example 6.—Sum the series $1 . 2 + 2 . 3 + 3 . 4 + \ldots + n(n+1)$.

The rth term of this series is

$$r(r+1) = r^2 + r.$$

$$\therefore \text{ Sum of series} = \sum_1^n (r^2 + r) = \sum_1^n r^2 + \sum_1^n r$$

$$= S_2 + S_1$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{6}(2n+1+3) = \frac{n(n+1)(n+2)}{3}.$$

Example 7.—Find the sum to n terms of the series

$$1.4.7 + 4.7.10 + 7.10.13 + \dots.$$

The rth term of the series is

$$[1 + 3(r-1)][4 + 3(r-1)][7 + 3(r-1)]$$

$$= (3r-2)(3r+1)(3r+4) = (9r^2 - 3r - 2)(3r + 4)$$

$$= 27r^3 + 27r^2 - 18r - 8.$$

Hence the sum of the series to n terms is

$$\sum_1^n (27r^3 + 27r^2 - 18r - 8)$$

$$= 27S_3 + 27S_2 - 18S_1 - 8n$$

$$= 27\left[\frac{n(n+1)}{2}\right]^2 + \frac{27n(n+1)(2n+1)}{6} - \frac{18n(n+1)}{2} - 8n$$

$$= \frac{n}{4}\{(27n^3 + 54n^2 + 27n) + (36n^2 + 54n + 18) - (36n + 36) - 32\}$$

$$= \frac{n}{4}\{27n^3 + 90n^2 + 45n - 50\}.$$

8. Summation by use of partial fractions.

In this method the terms in the given series to be summed are in fractional form, and the method of procedure is to find the rth term of the series and express it in its partial fractions. Then r is given the values $1, 2, 3, \dots, n$, and the sum of the terms thus obtained is found.

Example 8.—Find the sum of n terms, and also of an infinite number of terms of the series

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots.$$

Let S_n be the sum of n terms of the given series.

The rth term of the series is

$$\frac{1}{(3r-2)(3r+1)} = \frac{1}{3}\left(\frac{1}{3r-2} - \frac{1}{3r+1}\right). \quad \left(\begin{array}{c}\text{Partial fractions}\\ \text{by inspection.}\end{array}\right)$$

$$\begin{array}{ll} r=1 & \text{1st term} = \frac{1}{3}(1-\frac{1}{4}). \\ r=2 & \text{2nd term} = \frac{1}{3}(\frac{1}{4}-\frac{1}{7}). \\ r=3 & \text{3rd term} = \frac{1}{3}(\frac{1}{7}-\frac{1}{10}). \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ r=n & n\text{th term} = \frac{1}{3}\left(\frac{1}{3n-2} - \frac{1}{3n+1}\right). \end{array}$$

Adding these results, $S_n = \frac{1}{3}\left(1 - \frac{1}{3n+1}\right) = \frac{n}{3n+1}$.

Using $n = \infty$ in the first of these results, $S_\infty = \frac{1}{3}$.

Example 9.—Find the sum of n terms of the series

$$\frac{5}{1.2}(\tfrac{1}{3}) + \frac{7}{2.3}(\tfrac{1}{3})^2 + \frac{9}{3.4}(\tfrac{1}{3})^3 + \dots.$$

Let S_n be the required sum.

The rth term of the series is $\dfrac{5+2(r-1)}{r(r+1)}(\tfrac{1}{3})^r = \dfrac{2r+3}{r(r+1)}(\tfrac{1}{3})^r$

$$= \frac{3(r+1)-r}{r(r+1)}(\tfrac{1}{3})^r$$

$$= \left(\frac{3}{r} - \frac{1}{r+1}\right)(\tfrac{1}{3})^r.$$

$$\begin{array}{ll} r=1 & \text{1st term} = (3-\frac{1}{2})\frac{1}{3}. \\ r=2 & \text{2nd term} = (\frac{3}{2}-\frac{1}{3})(\frac{1}{3})^2. \\ r=3 & \text{3rd term} = (\frac{3}{3}-\frac{1}{4})(\frac{1}{3})^3. \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ r=n & n\text{th term} = \left(\frac{3}{n} - \frac{1}{n+1}\right)(\frac{1}{3})^n. \end{array}$$

$$\therefore S_n = \tfrac{1}{3} \times 3\left[1 + \frac{(\frac{1}{3})}{2} + \frac{(\frac{1}{3})^2}{3} + \dots + \frac{(\frac{1}{3})^{n-1}}{n}\right]$$

$$- \left[\frac{(\frac{1}{3})}{2} + \frac{(\frac{1}{3})^2}{3} + \frac{(\frac{1}{3})^3}{4} + \dots + \frac{(\frac{1}{3})^n}{n+1}\right]$$

$$= 1 - \frac{(\frac{1}{3})^n}{n+1} = 1 - \frac{1}{(n+1)3^n}.$$

9. Method of induction.

This method is exemplified in the proof of the binomial theorem for a positive integral index. It assumes the result to be true for a

quantity n and then proves that, if this be true, the theorem must be also true for the quantity $(n+1)$. Next, the given theorem is proved for $n = 1$, 2, and 3, and hence must be true for $n = 4$, $n = 5$, etc.

Example 10 (L.U.).—By induction, or otherwise, show that the sum to n terms of the series

$$\sum_1^n \tan^{-1}\frac{1}{2r^2} \text{ is } \tan^{-1}(2n+1) - \pi/4.$$

Assume that $$\sum_1^n \tan^{-1}\frac{1}{2r^2} = \tan^{-1}(2n+1) - \pi/4. \quad . \; . \; . \; . \; . \; . \quad \text{(i)}$$

Adding $\tan^{-1}\dfrac{1}{2(n+1)^2}$ to each side,

$$\sum_1^n \tan^{-1}\frac{1}{2r^2} + \tan^{-1}\frac{1}{2(n+1)^2} = \tan^{-1}(2n+1) - \frac{\pi}{4} + \tan^{-1}\frac{1}{2(n+1)^2}. \quad \text{(ii)}$$

Let $\theta_1 = \tan^{-1}(2n+1)$, and $\theta_2 = \tan^{-1} 1/\{2(n+1)^2\}$.

$$\therefore \tan\theta_1 = 2n+1, \; \tan\theta_2 = 1/\{2(n+1)^2\}.$$

$$\therefore \tan(\theta_1+\theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\tan\theta_2} = \frac{2n+1+\dfrac{1}{2(n+1)^2}}{1 - \dfrac{2n+1}{2(n+1)^2}}$$

$$= \frac{2(2n+1)(n^2+2n+1)+1}{2n^2+4n+2-2n-1} = \frac{4n^3+10n^2+8n+3}{2n^2+2n+1}$$

$$= 2n+3 \text{ (by ordinary division).}$$

$$\therefore \theta_1+\theta_2 = \tan^{-1}(2n+1) + \tan^{-1} 1/\{2(n+1)^2\} = \tan^{-1}(2n+3).$$

Hence equation (ii) becomes

$$\sum_1^{n+1} \tan^{-1}\frac{1}{2r^2} = \tan^{-1}(2n+3) - \pi/4,$$

which is the result of replacing n by $(n+1)$ in (i). Hence, if result (i) be true for n, it is also true for $(n+1)$. . . . . (A)

In (i), using $n = 1$, L.H.S. $= \tan^{-1}\frac{1}{2}$, R.H.S. $= \tan^{-1}3 - \pi/4$.

But $$\alpha - \beta = \tan^{-1}\left(\frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}\right). \quad . \; . \; . \; . \; . \; . \quad \text{(iii)}$$

$$\therefore \text{R.H.S.} = \tan^{-1}\frac{3-1}{1+3\,.\,1} \qquad \left(\tan\frac{\pi}{4} = 1\right)$$

$$= \tan^{-1}\tfrac{1}{2}.$$

$$\therefore \text{ equation (i) is true for } n = 1.$$

Using $n = 2$ in (i), L.H.S. $= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{8}$, and R.H.S. $= \tan^{-1}5 - \pi/4$.

Since $$\alpha + \beta = \tan^{-1}\left\{\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\right\},$$

$$\text{L.H.S.} = \tan^{-1}\left(\frac{\frac{1}{2} + \frac{1}{8}}{1 - \frac{1}{16}}\right) = \tan^{-1}\tfrac{10}{15} = \tan^{-1}\tfrac{2}{3},$$

and $$\text{R.H.S.} = \tan^{-1}\left\{\frac{5 - 1}{1 + 5\,.\,1}\right\} = \tan^{-1}\tfrac{2}{3}.$$

$$\therefore \text{ equation (i) is true for } n = 2.$$

Since equation (i) is true for $n = 2$, by statement (A) it must be true for $n = 3$; since it is true for $n = 3$, it must, by statement (A), be true for $n = 4$; and so on for all positive integral values of n,

$$\text{i.e. } \sum_1^n \tan^{-1}\frac{1}{2r^2} = \tan^{-1}(2n + 1) - \pi/4.$$

10. Summation of sine and cosine series (using complex numbers).

If it is possible to find the sum of the series $A_0 + A_1x + A_2x^2 + \ldots$, where all the A's are constants, then it is also possible to find the sum of each of the series

$$A_0 + A_1x\cos\theta + A_2x^2\cos 2\theta + \ldots,$$

and $$A_1x\sin\theta + A_2x^2\sin 2\theta + \ldots .$$

For, denoting the sum of the first series by C and that of the second series by S,

$$\begin{aligned} C + iS &= A_0 + A_1x(\cos\theta + i\sin\theta) + A_2x^2(\cos 2\theta + i\sin 2\theta) + \ldots \\ &= A_0 + A_1xe^{i\theta} + A_2x^2e^{2i\theta} + \ldots \\ &= A_0 + A_1y + A_2y^2 + \ldots, \text{ where } y = xe^{i\theta}. \end{aligned}$$

The sum of this latter series can be found, since it is the same as the original series with x replaced by y.

When the sum of this series $A_0 + A_1y + A_2y^2 + \ldots$ has been found, it can be put in the form $(a + ib)$ and, by equating real and imaginary quantities, the values of the series C and S can be obtained.

Example 11.—Sum the series

(*a*) $x\sin\theta - x^2\sin 2\theta + x^3\sin 3\theta - x^4\sin 4\theta + \ldots$ to n terms;

(*b*) $1 + x\cos\theta + \dfrac{x^2}{2!}\cos 2\theta + \dfrac{x^3}{3!}\cos 3\theta + \ldots \to \infty$.

Using the following values in this identity:

$$n=0 \qquad 1=2A, \quad \therefore A=\tfrac{1}{2}.$$
$$n=-1 \qquad 1=-B, \quad \therefore B=-1.$$
$$n=-2 \qquad 1=2C, \quad \therefore C=\tfrac{1}{2}.$$

$$\therefore \ n\text{th term} = \tfrac{1}{2}(-1)^{n+1}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right)x^n.$$

$$n=1 \quad \text{1st term} = \tfrac{1}{2}\left[x-2\left(\frac{x}{2}\right)+\frac{1}{3}x\right].$$

$$n=2 \quad \text{2nd term} = \tfrac{1}{2}\left[-\frac{x^2}{2}+2\left(\frac{x^2}{3}\right)-\frac{1}{4}x^2\right].$$

$$n=3 \quad \text{3rd term} = \tfrac{1}{2}\left[\frac{x^3}{3}-2\left(\frac{x^3}{4}\right)+\frac{x^3}{5}\right].$$

$\therefore$ Sum of series

$$=\tfrac{1}{2}\left[\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right)-2\left(\frac{x}{2}-\frac{x^2}{3}+\frac{x^3}{4}-\frac{x^4}{5}+\ldots\right) + \left(\frac{x}{3}-\frac{x^2}{4}+\frac{x^3}{5}-\frac{x^4}{6}+\ldots\right)\right]$$

$$=\tfrac{1}{2}\left[\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right)+\frac{2}{x}\left(-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right)+\frac{1}{x^2}\left(\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right)\right]$$

$$=\tfrac{1}{2}\left[\log_e(1+x)+\frac{2}{x}\left\{\log_e(1+x)-x\right\}+\frac{1}{x^2}\left\{\log_e(1+x)-x+\frac{x^2}{2}\right\}\right]$$

$$=\tfrac{1}{2}\left[\log_e(1+x)\left\{1+\frac{2}{x}+\frac{1}{x^2}\right\}-2-\frac{1}{x}+\tfrac{1}{2}\right]$$

$$=\frac{(x+1)^2}{2x^2}\log_e(1+x)-\frac{3x+2}{4x}.$$

EXAMPLES ON CHAPTER III

All the following questions are taken from London University papers.

1. Resolve $\dfrac{1+3x+12x^2}{(1+x)^2(1+4x)^2}$ into partial fractions and state the ranges of values of x for which the function may be expanded (i) in ascending, (ii) in descending powers of x. (See hint for Example 1.)

Find the coefficients of x^{2n} and x^{2n+1} in case (i), and the coefficients of x^{-2n} and $x^{-(2n+1)}$ in case (ii).

[*Hint.*—$(a+x)^n = x^n(1+a/x)^n$

$$= x^n\left\{1+n\cdot\frac{a}{x}+\frac{n(n-1)}{2!}\left(\frac{a}{x}\right)^2+\ldots\right\}$$

for expansion in descending powers of x.]

2. Express $\dfrac{1}{(1+x)^3(1-x)}$ in partial fractions.

If $|x| < 1$, and the above expression be expanded in a series of ascending powers of x, show that the ratio of the coefficient of x^{2n+1} to that of x^{2n} is $-(n+2)/(n+1)$.

3. Resolve $\dfrac{60x - 26 - 25x^2}{(5x-1)^2(2x-1)}$ into partial fractions.

Find the restrictions that must be imposed on x in order that

(i) The expression may be expanded as an infinite series in ascending powers of x.

(ii) The expression may be expanded as an infinite series in descending powers of x.

(iii) The expression may be expanded as an infinite series containing both ascending and descending powers of x.

In (i) find the coefficient of x^n; in (ii) find the coefficient of $1/x^n$; in (iii) find the coefficients of x^n and $1/x^n$.

[See hint for Example 1.]

4. Express $\dfrac{3x-1}{(1-x+x^2)(2+x)}$ in terms of partial fractions.

Find the coefficient of x^n in the expansion of this expression in positive powers of x. State the range of values of x for which the expansion is valid.

5. State the expansion of $\log_e(1+x)$ in ascending powers of x and prove that the series is convergent if $|x| < 1$.

Show that the sum of the infinite series whose rth term is

$$\frac{4r-1}{2r(2r-1)}\cdot\frac{1}{2^{2r}} \text{ is } \log_e 2 - \tfrac{1}{4}\log_e 3.$$

6. If u_r denotes the rth term of an infinite series, find the sum of the series when

$$\text{(i) } u_r = \frac{3r+1}{(2r+1)!}, \quad \text{(ii) } u_r = \frac{1}{2r(2r-1)}\cdot\frac{1}{2^{2r}}.$$

7. Assuming the logarithmic series and conditions for convergency, obtain an expansion for $\log_e x$ in ascending powers of $(x-1)/(x+1)$, giving the range of values of x for which it is valid.

Prove that for this range of values of x, the sum of the infinite series whose rth term is $\dfrac{r}{4r^2-1}\left(\dfrac{x-1}{x+1}\right)^{2r-1}$ is $\dfrac{(x^2+1)\log_e x - x^2 + 1}{4(x-1)^2}$.

8. Sum to n terms:

$$\text{(i) } \sinh\theta + \sinh 2\theta + \sinh 3\theta + \ldots .$$

$$\text{(ii) } x\sin\theta + 2x^2\sin 2\theta + 3x^3\sin 3\theta + \ldots ;$$

9. (i) Show that the sum to infinity of the series whose rth term is

$$\cos^r\beta\sin(\alpha + r\beta) \text{ is } \cot\beta\cos\alpha.$$

(ii) Sum to infinity the series whose rth term is $\cos^r\theta\sin r\theta/r!$.

10. (i) Sum to n terms the series whose rth term is $\dfrac{1}{(2r-1)(2r+1)(2r+3)}$, and find its sum to infinity.

(ii) Find the sum to infinity of the series whose rth terms are

$$\frac{4r^2+3r+1}{r!},\quad \frac{\sin r\theta}{r!}.$$

11. Sum to infinity the series whose nth terms are

$$\text{(i)}\ \frac{1}{n(n+1)},\quad \text{(ii)}\ \frac{n\cos n\theta}{(n+1)!}.$$

Sum to n terms the series whose rth term is $\cos\{\alpha+(r-1)\beta\}$. [*Hint.*—Multiply through series by $\sin\beta/2$ and then express each product as the difference of two sines.]

12. (i) Show that $4\sin^3\phi = 3\sin\phi - \sin 3\phi$.

Sum to n terms, and to infinity, the series whose mth term is $\dfrac{\sin^3(3^{m-1}\theta)}{3^{m-1}}$.

(ii) Sum the infinite series

$$\cos\theta - \tfrac{1}{2}\cos 3\theta + \frac{1.3}{2.4}\cos 5\theta - \frac{1.3.5}{2.4.6}\cos 7\theta + \dots$$

where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

13. (i) Show that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n2^n}\cos\frac{n\pi}{3} = \log_e 2 - \tfrac{1}{2}\log_e 3$.

(ii) Find the sum of the series $1 + \displaystyle\sum_{r=1}^{\infty}\frac{\cos 2r\theta}{(2r)!}$.

14. (i) State the expansion of $\log_e(1+x)$ in ascending powers of x, and state the range of values of x for which the series is convergent.

Show that, if p and q are positive, then

$$\log_e\frac{p}{q} = 2\left[\frac{p-q}{p+q} + \tfrac{1}{3}\left(\frac{p-q}{p+q}\right)^3 + \tfrac{1}{5}\left(\frac{p-q}{p+q}\right)^5 + \dots\right].$$

(ii) Show that the sum of the infinite series whose rth term is

$$\frac{10r+1}{2r(2r-1)(2r+1)}\cdot\frac{1}{2^{2r}}\ \text{is}\ 2 - \log_e 2 - \tfrac{3}{4}\log_e 3.$$

15. Find the sum to infinity of the series whose rth terms are

$$\text{(i)}\ \frac{1}{r(r+1)(r+3)},\quad \text{(ii)}\ \frac{2r^2-4r+3}{r!},\quad \text{(iii)}\ \frac{\cos r\theta}{(r+1)!}.$$

16. (i) Resolve the expression $\dfrac{r+1}{(r+2)(r+3)(r+4)}$ into partial fractions, and hence find the sum of the first n terms of the series of which this is the rth term.

(ii) Sum to infinity the series whose rth term is $\dfrac{4r-1}{2r(2r-1)}\cdot\dfrac{1}{3^{2r}}$.

CHAPTER IV

Differentiation

1. In its initial stages differentiation is largely a matter of finding limiting values, where the variable (δx) approaches zero, and to begin this chapter a few examples on limiting values will be taken. In most of these examples it will be seen that the problem involves the theory of Chap. II and necessitates the use of the expansions developed there.

Example 1 (L.U.).—Find (i) $\underset{h\to 0}{\text{Lt}}\left\{\dfrac{\sin(x+h)+\sin(x-h)-2\sin x}{h^2}\right\}$,

(ii) $\underset{h\to 0}{\text{Lt}}\left\{\dfrac{\log_e(x+h)+\log_e(x-h)-2\log_e x}{h^2}\right\}$.

(i) $$\underset{h\to 0}{\text{Lt}}\frac{\sin(x+h)+\sin(x-h)-2\sin x}{h^2}=\underset{h\to 0}{\text{Lt}}\frac{2\sin x\cos h-2\sin x}{h^2}$$

$$=\underset{h\to 0}{\text{Lt}}\frac{2\sin x(\cos h-1)}{h^2}=\underset{h\to 0}{\text{Lt}}\frac{2\sin x\left\{\left(1-\frac{h^2}{2!}+\frac{h^4}{4!}-\ldots\right)-1\right\}}{h^2}$$

$$=\underset{h\to 0}{\text{Lt}}\,2\sin x\left\{-\frac{h^2}{2!}+\frac{h^4}{4!}-\ldots\right\}\frac{1}{h^2}=\underset{h\to 0}{\text{Lt}}\,2\sin x\left\{-\frac{1}{2!}+\frac{h^2}{4!}-\ldots\right\}$$

$$=2\sin x\left(-\frac{1}{2!}\right)=-\sin x.$$

(ii) $$\underset{h\to 0}{\text{Lt}}\frac{\log_e(x+h)+\log_e(x-h)-2\log_e x}{h^2}=\underset{h\to 0}{\text{Lt}}\frac{\log_e\dfrac{x^2-h^2}{x^2}}{h^2}$$

$$=\underset{h\to 0}{\text{Lt}}\frac{\log_e\left(1-\frac{h^2}{x^2}\right)}{h^2}=\underset{h\to 0}{\text{Lt}}\frac{\left(-\frac{h^2}{x^2}-\frac{h^4}{2x^4}-\ldots\right)}{h^2}$$

$$=\underset{h\to 0}{\text{Lt}}\left(-\frac{1}{x^2}-\frac{h^2}{2x^4}-\ldots\right)=-\frac{1}{x^2}.$$

Example 2 (L.U.).—If $y = \dfrac{A}{m^2}\left(\dfrac{\sin mx}{\sin ml} - \dfrac{x}{l}\right)$, find the limit to which y tends as $m \to 0$.

$$\underset{m\to 0}{\text{Lt}} \frac{A}{m^2}\left(\frac{\sin mx}{\sin ml} - \frac{x}{l}\right) = \underset{m\to 0}{\text{Lt}} \frac{A}{m^2}\left\{\sin mx(\sin ml)^{-1} - \frac{x}{l}\right\}$$

$$= \underset{m\to 0}{\text{Lt}} \frac{A}{m^2}\left\{\left[mx - \frac{(mx)^3}{3!} + \ldots\right]\left[ml - \frac{(ml)^3}{3!} + \ldots\right]^{-1} - \frac{x}{l}\right\}$$

(using the series for $\sin\theta$)

$$= \underset{m\to 0}{\text{Lt}} \frac{A}{m^2}\left\{mx\left(1 - \frac{m^2x^2}{3!} + \ldots\right)\frac{1}{ml}\left[1 - \left(\frac{m^2l^2}{3!} - \ldots\right)\right]^{-1} - \frac{x}{l}\right\}$$

$$= \underset{m\to 0}{\text{Lt}} \frac{A}{m^2}\left\{\frac{x}{l}\left(1 - \frac{m^2x^2}{3!} + \ldots\right)\left(1 + \frac{m^2l^2}{3!} + \ldots\right) - \frac{x}{l}\right\}$$

$$= \underset{m\to 0}{\text{Lt}} \frac{Ax/l}{m^2}\left\{1 + \frac{m^2l^2}{3!} - \frac{m^2x^2}{3!} + \text{terms in } m^4, \text{etc.} - 1\right\}$$

$$= \underset{m\to 0}{\text{Lt}} A\frac{x}{l}\left\{\frac{l^2}{3!} - \frac{x^2}{3!} + \text{terms in } m^2, \text{etc.}\right\}$$

$$= \frac{Ax}{6l}\left\{l^2 - x^2\right\}.$$

Example 3.—Find the following limits:

$$\text{(i)}\quad \underset{x\to 0}{\text{Lt}} \frac{\cosh x - 1}{x^2},$$

$$\text{(ii)}\quad \underset{x\to \infty}{\text{Lt}} \frac{2x^2 + 3x + 2}{3x^2 + x - 1},$$

$$\text{(iii)}\quad \underset{x\to 0}{\text{Lt}} \frac{2\cos x - 2 + x^2}{x^4}.$$

(i) Using the series for $\cosh x$,

$$\underset{x\to 0}{\text{Lt}} \frac{\cosh x - 1}{x^2} = \underset{x\to 0}{\text{Lt}} \frac{(1 + x^2/2! + x^4/4! + \ldots) - 1}{x^2}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{x^2/2! + x^4/4! + \ldots}{x^2}$$

$$= \underset{x\to 0}{\text{Lt}} (1/2! + x^2/4! + \ldots)$$

$$= \tfrac{1}{2}.$$

(ii)

$$\underset{x\to \infty}{\text{Lt}} \frac{2x^2 + 3x + 2}{3x^2 + x - 1} = \underset{x\to \infty}{\text{Lt}} \frac{2 + 3/x + 2/x^2}{3 + 1/x - 1/x^2}$$

$$= \frac{\underset{x\to \infty}{\text{Lt}} (2 + 3/x + 2/x^2)}{\underset{x\to \infty}{\text{Lt}} (3 + 1/x - 1/x^2)}$$

$$= \tfrac{2}{3}.$$

(iii) $$\underset{x\to 0}{\mathrm{Lt}}\frac{2\cos x-2+x^2}{x^4}=\underset{x\to 0}{\mathrm{Lt}}\frac{2(1-x^2/2!+x^4/4!-x^6/6!+\ldots)-2+x^2}{x^4}$$
$$=\underset{x\to 0}{\mathrm{Lt}}\frac{2x^4/4!-2x^6/6!+\ldots}{x^4}$$
$$=\underset{x\to 0}{\mathrm{Lt}}\left(\frac{1}{3.4}-\frac{2x^2}{6!}+\ldots\right)$$
$$=\frac{1}{3.4}=\frac{1}{12}.$$

2. Revision notes on differentiation.

The first differential coefficient or derivative of y with respect to x is $\underset{\delta x\to 0}{\mathrm{Lt}}\frac{\delta y}{\delta x}$, where δy is a small increase in the dependent variable y, corresponding to a small increment δx in x, and y is some function of x. This differential coefficient is denoted by any one of the symbols $\frac{dy}{dx}$, $\frac{d}{dx}\{f(x)\}$, $f'(x)$, Dy, where $y=f(x)$. Hence the definition of the first differential coefficient of y with respect to x is

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\mathrm{Lt}}\frac{\delta y}{\delta x}.$$

If $y=f(x)$, it can be readily shown that

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\mathrm{Lt}}\frac{f(x+\delta x)-f(x)}{\delta x},$$

and this result is known as the *derived definition* of the differential coefficient.

When obtaining a differential coefficient from *first principles*, it is necessary to work from one of the above two definitions without using any of the theorems on differential calculus that follow, and it is usually quicker to use the derived definition.

The following theorems have been dealt with at the Intermediate stage and are merely stated:

Theorem 1.—The derivative of a constant is zero.

Theorem 2.—$\frac{d}{dx}(x^n)=nx^{n-1}$ for all values of n.

Theorem 3.—The differential coefficient of a sum is equal to the sum of the differential coefficients.

Theorem 4.—If y be a function of z, where z is a function of x, then $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$ (function of a function theorem).

Theorem 5.—If u and v be functions of x, then

$$\text{(i)} \quad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx};$$

$$\text{(ii)} \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Theorem 6.—If a be any constant, then

$$\frac{d}{dx}\{af(x)\} = a\frac{d}{dx}\{f(x)\}.$$

Theorem 7.—Using the function of a function theorem, where a and b are constants,

$$\frac{d}{dx}\{f(ax+b)\} = a\frac{d}{dz}\{f(z)\}, \text{ where } z = ax + b.$$

In particular $\frac{d}{dx}[(ax+b)^n] = a\frac{d}{dz}(z^n)$, where $z = ax + b$

$$= anz^{n-1} = an(ax+b)^{n-1}.$$

Theorem 8.—
$$\left.\begin{aligned} \frac{d}{dx}(\sin x) &= \cos x \\ \frac{d}{dx}(\cos x) &= -\sin x \end{aligned}\right\}, \text{ where } x \text{ is in } \textit{radians}.$$

Theorem 9.—The slope of the curve $y = f(x)$ at (x, y) is $\frac{dy}{dx}$.

Theorem 10.—
$$\frac{dx}{dy} = 1\Big/\frac{dy}{dx}.$$

3. Derivatives of remaining trigonometric functions.

N.B.—x is in *radians* throughout.

(i) To find $\frac{d}{dx}(\tan x)$ *from first principles.*

$$\frac{d}{dx}(\tan x) = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{\tan(x+\delta x) - \tan x}{\delta x} \text{ (using derived definition)}$$

$$= \underset{\delta x \to 0}{\mathrm{Lt}} \frac{\dfrac{\sin(x+\delta x)}{\cos(x+\delta x)} - \dfrac{\sin x}{\cos x}}{\delta x}$$

$$= \underset{\delta x \to 0}{\mathrm{Lt}} \frac{\sin(x+\delta x)\cos x - \sin x \cos(x+\delta x)}{\delta x \cos(x+\delta x)\cos x}$$

$$= \underset{\delta x \to 0}{\mathrm{Lt}} \frac{1}{\cos(x+\delta x)\cos x} \cdot \frac{\sin \delta x}{\delta x}$$

$$= \underset{\delta x \to 0}{\mathrm{Lt}} \frac{1}{\cos(x+\delta x)\cos x} \underset{\delta x \to 0}{\mathrm{Lt}} \frac{\sin \delta x}{\delta x} \text{ (limit of product).}$$

But $\underset{\delta x \to 0}{\mathrm{Lt}} \dfrac{\sin \delta x}{\delta x} = 1$ and $\underset{\delta x \to 0}{\mathrm{Lt}} \dfrac{1}{\cos(x+\delta x)\cos x} = \dfrac{1}{\cos^2 x}$.

$$\therefore \frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x.$$

(ii) To find $\dfrac{d}{dx}(\sec x)$ [not from first principles].

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}(z^{-1}), \text{ where } z = \cos x, \text{ and } \therefore \frac{dz}{dx} = -\sin x$$

$$= \frac{d}{dz}(z^{-1})\frac{dz}{dx} \text{ (function of function)}$$

$$= (-1)z^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x}.$$

$$\therefore \frac{d}{dx}(\sec x) = \sec x \tan x.$$

(iii) To find $\dfrac{d}{dx}(\operatorname{cosec} x)$.

$$\frac{d}{dx}(\operatorname{cosec} x) = \frac{d}{dx}(z^{-1}), \text{ where } z = \sin x, \text{ and } \therefore \frac{dz}{dx} = \cos x$$

$$= \frac{d}{dz}(z^{-1})\frac{dz}{dx} = (-1)z^{-2}\cos x.$$

$$\therefore \frac{d}{dx}(\operatorname{cosec} x) = -\frac{\cos x}{\sin^2 x} = -\operatorname{cosec} x \cot x.$$

(iv) To find $\dfrac{d}{dx}(\cot x)$.

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$= \frac{\sin x \dfrac{d}{dx}(\cos x) - \cos x \dfrac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}.$$

$$\therefore \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x.$$

Using the function of a function theorem the following results are obtained:

$$\frac{d}{dx}\{\sin(ax+b)\} = a\cos(ax+b);$$

$$\frac{d}{dx}\{\cos(ax+b)\} = -a\sin(ax+b);$$

$$\frac{d}{dx}\{\tan(ax+b)\} = a\sec^2(ax+b);$$

$$\frac{d}{dx}\{\sec(ax+b)\} = a\sec(ax+b)\tan(ax+b);$$

$$\frac{d}{dx}\{\operatorname{cosec}(ax+b)\} = -a\operatorname{cosec}(ax+b)\cot(ax+b);$$

$$\frac{d}{dx}\{\cot(ax+b)\} = -a\operatorname{cosec}^2(ax+b).$$

N.B.—When the angle x is given in degrees, it must be converted to radians before the above results can be applied.

Thus $$\frac{d}{dx}(\tan x^\circ) = \frac{d}{dx}\left(\tan\frac{x\pi}{180}\right) = \frac{\pi}{180}\sec^2\frac{x\pi}{180}$$

$$= \frac{\pi}{180}\sec^2 x^\circ.$$

Example 4.—Find from first principles $\frac{d}{dx}(2x \sin 2x)$.

Using the derived definition,

$$\frac{d}{dx}(2x \sin 2x) = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{2(x + \delta x)\sin(2x + 2\delta x) - 2x \sin 2x}{\delta x}$$

$$= \underset{\delta x \to 0}{\mathrm{Lt}} \frac{2x\{\sin(2x + 2\delta x) - \sin 2x\} + 2\delta x \sin(2x + 2\delta x)}{\delta x}$$

$$= \underset{\delta x \to 0}{\mathrm{Lt}} \frac{4x \cos(2x + \delta x)\sin \delta x + 2\delta x \sin(2x + 2\delta x)}{\delta x}$$

$$= \underset{\delta x \to 0}{\mathrm{Lt}} \left\{4x \cos(2x + \delta x)\frac{\sin \delta x}{\delta x}\right\} + \underset{\delta x \to 0}{\mathrm{Lt}}\ 2\sin(2x + 2\delta x)$$

(limit of sum)

$$= \underset{\delta x \to 0}{\mathrm{Lt}} \{4x \cos(2x + \delta x)\}\ \underset{\delta x \to 0}{\mathrm{Lt}} \frac{\sin \delta x}{\delta x} + 2\sin 2x$$

(limit of product).

But $\underset{\delta x \to 0}{\mathrm{Lt}} \frac{\sin \delta x}{\delta x} = 1$, and $\underset{\delta x \to 0}{\mathrm{Lt}} \{4x \cos(2x + \delta x)\} = 4x \cos 2x$.

$$\therefore \frac{d}{dx}(2x \sin 2x) = 4x \cos 2x + 2 \sin 2x.$$

4. Derivatives of the exponential and logarithmic functions.

(i) To find $\frac{d}{dx}(e^x)$.

$$\frac{d}{dx}(e^x) = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \to \infty\right)$$

$$= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \ldots \to \infty$$

$$= 1 + x + \frac{x^2}{2!} + \ldots \to \infty.$$

$$\therefore \frac{d}{dx}(e^x) = e^x.$$

Aliter (from first principles).

$$\frac{d}{dx}(e^x) = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{e^{x+\delta x} - e^x}{\delta x} = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{e^x(e^{\delta x} - 1)}{\delta x}$$

$$= \underset{\delta x \to 0}{\mathrm{Lt}} \frac{e^x\{(1 + \delta x + (\delta x)^2/2! + \ldots) - 1\}}{\delta x}$$

$$= \underset{\delta x \to 0}{\mathrm{Lt}}\ e^x\left\{1 + \frac{\delta x}{2!} + \ldots\right\} = e^x.$$

Using the function of a function theorem it follows that

$$\frac{d}{dx}(e^{ax+b}) = ae^{ax+b}.$$

(ii) To find $\frac{d}{dx}(a^x)$, where a is a constant.

Let $a^x = e^y$.

Taking logs to the base e,

$$x \log_e a = y, \ \therefore \ a^x = e^{x \log_e a}.$$

$$\therefore \ \frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x \log_e a}) = (\log_e a)e^{x \log_e a}$$

$$= a^x \log_e a.$$

The general result, using the function of a function theorem, is

$$\frac{d}{dx}(a^{bx+c}) = ba^{bx+c} \log_e a.$$

(iii) To find $\frac{d}{dx}(\log_e x)$.

Let $$\log_e x = y, \ \therefore \ x = e^y. \quad \dots \quad (1)$$

Differentiating (1) with respect to y,

$$\frac{dx}{dy} = \frac{d}{dy}(e^y) = e^y.$$

$$\therefore \ \frac{dy}{dx} = 1\Big/\frac{dx}{dy} = \frac{1}{e^y} = \frac{1}{x}.$$

$$\therefore \ \frac{d}{dx}(\log_e x) = \frac{1}{x}.$$

The general result is $\frac{d}{dx}\{\log_e (ax+b)\} = \frac{a}{ax+b}$.

N.B.—If the logarithm given is not to the base e, it is necessary to convert it to the base e before proceeding to differentiate, or to adopt the following procedure:

$$\text{Let } \log_a x = y, \ \therefore \ x = a^y. \quad \dots \quad (2)$$

Differentiating (2) with respect to x,

$$1 = \frac{d}{dx}(a^y) = \frac{d}{dy}(a^y) \cdot \frac{dy}{dx}$$

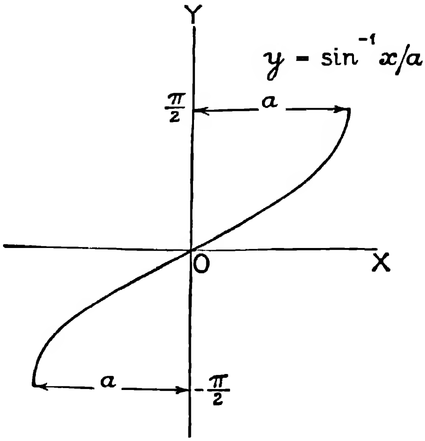
$$= a^y \log_e a \cdot \frac{dy}{dx}.$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\log_a x) = \frac{1}{a^y \log_e a} = \frac{1}{x \log_e a}.$$

5. Derivatives of inverse trigonometric functions.

N.B.—$\sin^{-1}x$ is the angle lying between $-\pi/2$ and $+\pi/2$ whose sine has the value x.

$\cos^{-1}x$ is the angle lying between 0 and π whose cosine has the value x.



$\tan^{-1}x$ is the angle lying between $-\pi/2$ and $+\pi/2$ whose tangent has the value x.

Any ambiguity in sign in the derivative result is decided by using the graph.

(i) To find $\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right)$.

Let $\qquad y = \sin^{-1} x/a, \quad \therefore x = a \sin y. \quad . \quad . \quad . \quad . \quad (3)$

Differentiating (3) with respect to x,

$$1 = \frac{d}{dx}(a \sin y) = \frac{d}{dy}(a \sin y)\frac{dy}{dx}.$$

$$\therefore\ 1 = a \cos y \frac{dy}{dx}.$$

$$\therefore\ \frac{dy}{dx} = \frac{1}{a \cos y} = \frac{\pm 1}{\sqrt{(a^2 \cos^2 y)}}$$

$$= \frac{\pm 1}{\sqrt{(a^2 - a^2 \sin^2 y)}} = \frac{\pm 1}{\sqrt{(a^2 - x^2)}}.$$

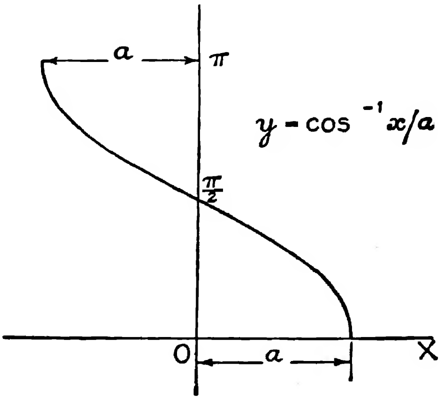
From the graph the slope of the curve $y = \sin^{-1} x/a$ is always positive, and therefore $\frac{dy}{dx}$ is always positive.

Thus $$\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{(a^2 - x^2)}}.$$

Using $a = 1$, $$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{(1 - x^2)}}.$$

(ii) To find $\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right)$.

Let $$y = \cos^{-1} x/a, \quad \therefore\ x = a \cos y. \quad . \quad . \qquad (4)$$



Differentiating (4) with respect to y,

$$\frac{dx}{dy} = -a \sin y.$$

$$\therefore \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{-1}{a \sin y}$$

$$= \frac{\pm 1}{\sqrt{(a^2 \sin^2 y)}} = \frac{\pm 1}{\sqrt{(a^2 - a^2 \cos^2 y)}} = \frac{\pm 1}{\sqrt{(a^2 - x^2)}}.$$

From the graph of $y = \cos^{-1} x/a$, the slope of the curve is always negative, and therefore dy/dx is always negative. Thus

$$\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right) = \frac{-1}{\sqrt{(a^2 - x^2)}}.$$

When $a = 1$, $$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{(1 - x^2)}}.$$

(iii) To find $\dfrac{d}{dx}\left(\tan^{-1}\dfrac{x}{a}\right)$.

Let $y = \tan^{-1}(x/a)$, $\therefore x = a \tan y$. . . . . (5)

Differentiating (5) with respect to x,

$$1 = \frac{d}{dx}(a \tan y) = \frac{d}{dy}(a \tan y) \cdot \frac{dy}{dx}$$

$$= a \sec^2 y \cdot \frac{dy}{dx}.$$

$$\therefore \frac{dy}{dx} = \frac{1}{a \sec^2 y} = \frac{1}{a(1 + \tan^2 y)} = \frac{a}{a^2 + a^2 \tan^2 y}$$

$$= \frac{a}{a^2 + x^2}.$$

$$\therefore \frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right) = \frac{a}{a^2 + x^2}.$$

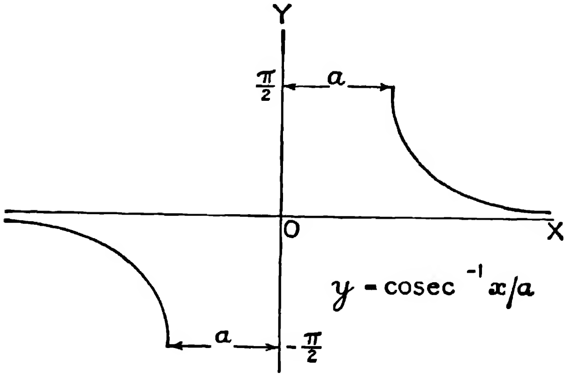
When $a = 1$, $$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$$

(iv) To find $\frac{d}{dx}\left(\operatorname{cosec}^{-1}\frac{x}{a}\right)$.

Let $\qquad y = \operatorname{cosec}^{-1} x/a, \quad \therefore\ x = a \operatorname{cosec} y. \quad . \quad . \quad . \quad (6)$

Differentiating (6) with respect to y,

$$\frac{dx}{dy} = -a \operatorname{cosec} y \cot y$$
$$= -x \cot y$$
$$= \pm x\sqrt{(\operatorname{cosec}^2 y - 1)}$$
$$= \pm x\sqrt{\left(\frac{x^2}{a^2} - 1\right)} = \pm\frac{x\sqrt{(x^2 - a^2)}}{a}.$$
$$\therefore\ \frac{dy}{dx} = \frac{\pm a}{x\sqrt{(x^2 - a^2)}}.$$



From the graph $\frac{dy}{dx}$ is always negative,

$$\therefore\ \frac{d}{dx}\left(\operatorname{cosec}^{-1}\frac{x}{a}\right) = \frac{-a}{x\sqrt{(x^2 - a^2)}}.$$

When $a = 1$, $\qquad \frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{(x^2 - 1)}}.$

(v) To find $\frac{d}{dx}\left(\sec^{-1}\frac{x}{a}\right)$.

Let $\qquad y = \sec^{-1} x/a, \quad \therefore\ x = a \sec y. \quad . \quad . \quad . \quad . \quad (7)$

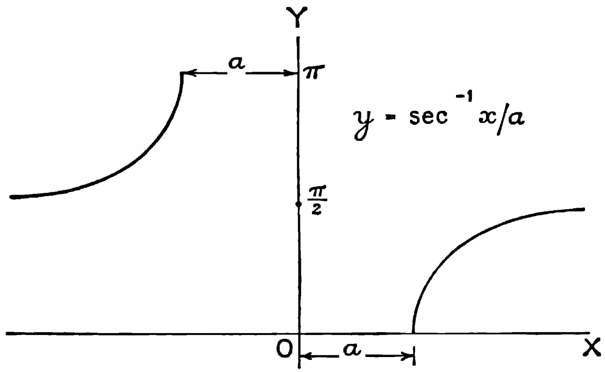
Differentiating (7) with respect to x,

$$1 = \frac{d}{dx}(a \sec y)$$

$$= a \sec y \tan y \frac{dy}{dx}.$$

$$\therefore \frac{dy}{dx} = \frac{1}{a \sec y \tan y} = \frac{a}{x \,.\, a \tan y} = \frac{\pm a}{x\sqrt{(a^2 \sec^2 y - a^2)}}$$

$$= \frac{\pm a}{x\sqrt{(x^2 - a^2)}}.$$



From the graph, $\frac{dy}{dx}$ is always positive.

$$\therefore \frac{d}{dx}\left(\sec^{-1}\frac{x}{a}\right) = \frac{a}{x\sqrt{(x^2 - a^2)}}.$$

When $a = 1$, $$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{(x^2 - 1)}}.$$

(vi) To find $\frac{d}{dx}\left(\cot^{-1}\frac{x}{a}\right)$.

Let $y = \cot^{-1} x/a, \quad \therefore x = a \cot y.$ . . . . (8)

Differentiating (8) with respect to y,

$$\frac{dx}{dy} = -a\,\text{cosec}^2 y = -a(1 + \cot^2 y)$$

$$= -a\left(1 + \frac{x^2}{a^2}\right) = -\frac{(a^2 + x^2)}{a}.$$

$$\therefore\ \frac{dy}{dx} = 1\Big/\frac{dx}{dy} = -\frac{a}{a^2 + x^2},$$

$$\text{i.e.}\ \frac{d}{dx}\left(\cot^{-1}\frac{x}{a}\right) = -\frac{a}{a^2 + x^2}.$$

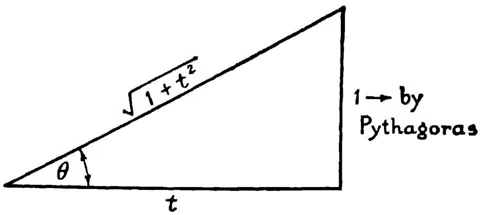
When $a = 1$, $$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1 + x^2}.$$

N.B.—Care must be taken to learn which of the general results has unity and which has a in the numerator.

Example 5.—Find (a) $\dfrac{d}{dx}\left(x \sin^{-1}\dfrac{x}{2}\right)$,

(b) $\dfrac{d}{dz}\{\tan^{-1}(2z + 3)\}$,

(c) $\dfrac{d}{dt}\left\{\sec^{-1}\left(\dfrac{\sqrt{(1 + t^2)}}{t}\right)\right\}$.



(a) $$\frac{d}{dx}\left(x \sin^{-1}\frac{x}{2}\right) = x\frac{d}{dx}\left(\sin^{-1}\frac{x}{2}\right) + \sin^{-1}\frac{x}{2}\frac{d}{dx}(x)$$

$$= \frac{x}{\sqrt{(4 - x^2)}} + \sin^{-1}\frac{x}{2}.$$

(b) $$\frac{d}{dz}\{\tan^{-1}(2z + 3)\} = \frac{d}{dz}(\tan^{-1} p),\ \text{where}\ p = 2z + 3$$

$$= \frac{d}{dp}(\tan^{-1} p)\frac{dp}{dz}$$

$$= \frac{1}{1 + p^2}\cdot 2 = \frac{2}{1 + (2z + 3)^2}$$

$$= \frac{2}{4z^2 + 12z + 10} = \frac{1}{2z^2 + 6z + 5}.$$

(*c*) In this case it is advisable to simplify the function before differentiating, by making use of the right-angled triangle shown on the previous page.

Let $$\theta = \sec^{-1}\frac{\sqrt{(1+t^2)}}{t}.$$

From the diagram $$\theta = \cot^{-1} t.$$

$$\therefore \frac{d\theta}{dt} = \frac{-1}{1+t^2}.$$

6. Derivatives of the hyperbolic functions.

(i) To find $\dfrac{d}{dx}(\sinh x)$.

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\frac{(e^x - e^{-x})}{2} = \tfrac{1}{2}[e^x - (-e^{-x})]$$
$$= \tfrac{1}{2}[e^x + e^{-x}] = \cosh x.$$

(ii) To find $\dfrac{d}{dx}(\cosh x)$.

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \tfrac{1}{2}(e^x - e^{-x}) = \sinh x.$$

(iii) To find $\dfrac{d}{dx}(\tanh x)$.

$$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$$
$$= \frac{\cosh x \dfrac{d}{dx}\sinh x - \sinh x \dfrac{d}{dx}\cosh x}{\cosh^2 x}$$
$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}.$$
$$\therefore \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x.$$

(iv) To find $\dfrac{d}{dx}(\operatorname{sech} x)$.

$$\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}(z^{-1}), \text{ where } z = \cosh x,$$
$$= \frac{d}{dz}(z^{-1}) \cdot \frac{dz}{dx} = -z^{-2}(\sinh x).$$
$$\therefore \frac{d}{dx}(\operatorname{sech} x) = \frac{-\sinh x}{\cosh^2 x} = -\operatorname{sech} x \tanh x.$$

(v) To find $\frac{d}{dx}(\operatorname{cosech} x)$.

$$\frac{d}{dx}(\operatorname{cosech} x) = \frac{d}{dx}(z^{-1}), \text{ where } z = \sinh x,$$

$$= \frac{d}{dz}(z^{-1})\frac{dz}{dx} = -z^{-2}(\cosh x).$$

$$\therefore \frac{d}{dx}(\operatorname{cosech} x) = \frac{-\cosh x}{\sinh^2 x} = -\operatorname{cosech} x \coth x.$$

(vi) To find $\frac{d}{dx}(\coth x)$.

$$\frac{d}{dx}(\coth x) = \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right) = \frac{\sinh x \frac{d}{dx}(\cosh x) - \cosh x \frac{d}{dx}(\sinh x)}{\sinh^2 x}$$

$$= \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x}.$$

$$\therefore \frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x.$$

Using the function of a function theorem, the following general results are obtained:

$$\frac{d}{dx}\{\sinh(ax+b)\} = a\cosh(ax+b);$$

$$\frac{d}{dx}\{\cosh(ax+b)\} = a\sinh(ax+b);$$

$$\frac{d}{dx}\{\tanh(ax+b)\} = a\operatorname{sech}^2(ax+b);$$

$$\frac{d}{dx}\{\operatorname{sech}(ax+b)\} = -a\operatorname{sech}(ax+b)\tanh(ax+b);$$

$$\frac{d}{dx}\{\operatorname{cosech}(ax+b)\} = -a\operatorname{cosech}(ax+b)\coth(ax+b);$$

$$\frac{d}{dx}\{\coth(ax+b)\} = -a\operatorname{cosech}^2(ax+b).$$

N.B.—The first three of these results are positive and the last three negative.

7. Derivatives of the inverse hyperbolic functions.

(i) To find $\dfrac{d}{dx}\left(\sinh^{-1}\dfrac{x}{a}\right)$.

Let $$y = \sinh^{-1}(x/a), \quad \therefore\ x = a\sinh y. \quad . \quad . \quad . \quad . \quad (9)$$

Differentiating (9) with respect to y,

$$\frac{dx}{dy} = a\cosh y = +\sqrt{(a^2 + a^2\sinh^2 y)},$$
(+ve root since $\cosh y$ positive)

$$\therefore\ \frac{dy}{dx} = \frac{1}{\sqrt{(a^2 + a^2\sinh^2 y)}} = \frac{1}{\sqrt{(a^2 + x^2)}},$$

$$\text{i.e. } \frac{d}{dx}\left(\sinh^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{(x^2 + a^2)}}.$$

(ii) To find $\dfrac{d}{dx}\left(\cosh^{-1}\dfrac{x}{a}\right)$.

Let $$y = \cosh^{-1}(x/a), \quad \therefore\ x = a\cosh y. \quad . \quad . \quad . \quad (10)$$

Differentiating (10) with respect to x,

$$1 = a\sinh y\,\frac{dy}{dx}.$$

$$\therefore\ \frac{dy}{dx} = \frac{1}{a\sinh y} = \frac{\pm 1}{\sqrt{(a^2\cosh^2 y - a^2)}}$$
(both signs admissible)

$$= \frac{\pm 1}{\sqrt{(x^2 - a^2)}},$$

$$\therefore\ \frac{d}{dx}\left(\cosh^{-1}\frac{x}{a}\right) = \frac{\pm 1}{\sqrt{(x^2 - a^2)}}.$$

(iii) To find $\dfrac{d}{dx}\left(\tanh^{-1}\dfrac{x}{a}\right)$.

Let $$y = \tanh^{-1}(x/a), \quad \therefore\ x = a\tanh y. \quad . \quad . \quad . \quad (11)$$

Differentiating (11) with respect to y,

$$\frac{dx}{dy} = a\operatorname{sech}^2 y = a(1 - \tanh^2 y)$$

$$= a(1 - x^2/a^2) = (a^2 - x^2)/a,$$

$$\therefore\ \frac{dy}{dx} = \frac{d}{dx}\left(\tanh^{-1}\frac{x}{a}\right) = \frac{a}{a^2 - x^2}$$

$(x^2 < a^2$ since $|\tanh\theta| < 1)$.

(iv) To find $\frac{d}{dx}\left(\operatorname{sech}^{-1}\frac{x}{a}\right)$.

Let $\quad y = \operatorname{sech}^{-1}(x/a), \quad \therefore\ x = a\operatorname{sech} y. \quad . \quad . \quad . \quad (12)$

Differentiating (12) with respect to x,

$$1 = -a\operatorname{sech} y\tanh y\,\frac{dy}{dx}.$$

$$\therefore\ \frac{dy}{dx} = \frac{-1}{a\operatorname{sech} y\tanh y} = \frac{-a}{x(a\tanh y)}$$

$$= \frac{\pm a}{x\sqrt{(a^2 - a^2\operatorname{sech}^2 y)}} = \frac{\pm a}{x\sqrt{(a^2 - x^2)}},$$

$$\therefore\ \frac{d}{dx}\left(\operatorname{sech}^{-1}\frac{x}{a}\right) = \frac{\pm a}{x\sqrt{(a^2 - x^2)}}$$

(both signs are admissible, and $x^2 < a^2$ since $\operatorname{sech}\theta < 1$).

(v) To find $\frac{d}{dx}\left(\operatorname{cosech}^{-1}\frac{x}{a}\right)$.

Let $\quad y = \operatorname{cosech}^{-1}(x/a), \quad \therefore\ x = a\operatorname{cosech} y. \quad . \quad . \quad (13)$

Differentiating (13) with respect to y,

$$\frac{dx}{dy} = -a\operatorname{cosech} y\coth y$$

$$= -x\coth y,$$

$$\therefore\ \frac{dy}{dx} = \frac{-a}{x(a\coth y)} = \frac{\pm a}{x\sqrt{(a^2 + a^2\operatorname{cosech}^2 y)}}$$

(both signs admissible).

$$\therefore\ \frac{d}{dx}\left(\operatorname{cosech}^{-1}\frac{x}{a}\right) = \frac{\pm a}{x\sqrt{(x^2 + a^2)}}.$$

(vi) To find $\dfrac{d}{dx}\left(\coth^{-1}\dfrac{x}{a}\right)$.

Let $y = \coth^{-1}(x/a)$, $\therefore\ x = a\coth y$. . . . (14)

Differentiating (14) with respect to x,

$$1 = -a\,\text{cosech}^2 y\,\frac{dy}{dx}.$$

$$\therefore\ \frac{dy}{dx} = \frac{-1}{a\,\text{cosech}^2 y} = \frac{-a}{a^2\,\text{cosech}^2 y}$$

$$= \frac{-a}{a^2\coth^2 y - a^2} = \frac{-a}{x^2 - a^2},$$

$$\therefore\ \frac{d}{dx}\left(\coth^{-1}\frac{x}{a}\right) = \frac{-a}{x^2 - a^2},$$

where $x^2 > a^2$ since $|\coth\theta| > 1$.

In particular, when $a = 1$ in these six results,

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{(x^2+1)}};$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{\pm 1}{\sqrt{(x^2-1)}} \quad (x^2 > 1);$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2} \quad (x^2 < 1);$$

$$\frac{d}{dx}(\text{sech}^{-1}x) = \frac{\pm 1}{x\sqrt{(1-x^2)}} \quad (x^2 < 1);$$

$$\frac{d}{dx}(\text{cosech}^{-1}x) = \frac{\pm 1}{x\sqrt{(x^2+1)}};$$

$$\frac{d}{dx}(\coth^{-1}x) = \frac{-1}{x^2-1} \quad (x^2 > 1).$$

Example 6 (L.U.).—Obtain from first principles the differential coefficient of $2\cos 3x$ with respect to x.

Find the x-derivatives of x^3e^{-3x}, $(x^3-1)/(x^3+1)$, and $\log_e(\tan\frac{1}{2}x)$.

Using the derived definition,

$$\frac{d}{dx}(2\cos 3x) = \underset{\delta x \to 0}{\text{Lt}} \frac{2\cos 3(x+\delta x) - 2\cos 3x}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{-4\sin\left(3x + \frac{3\delta x}{2}\right)\sin\frac{3\delta x}{2}}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \left\{-6\sin\left(3x + \frac{3\delta x}{2}\right). \frac{\sin\frac{3\delta x}{2}}{\frac{3\delta x}{2}}\right\}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \left\{-6\sin\left(3x + \frac{3\delta x}{2}\right)\right\} . \underset{\delta x \to 0}{\text{Lt}} \frac{\sin\frac{3\delta x}{2}}{\frac{3\delta x}{2}}$$

(Lt of product).

But $\underset{\delta x \to 0}{\text{Lt}} \dfrac{\sin\frac{3\delta x}{2}}{\frac{3\delta x}{2}} = 1$, and $\underset{\delta x \to 0}{\text{Lt}} \left\{-6\sin\left(3x + \frac{3\delta x}{2}\right)\right\} = -6\sin 3x$,

$$\therefore \frac{d}{dx}(2\cos 3x) = -6\sin 3x.$$

$$\frac{d}{dx}(x^2e^{-3x}) = e^{-3x}\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(e^{-3x})$$

$$= 2xe^{-3x} + x^2(-3e^{-3x})$$

$$= xe^{-3x}(2 - 3x).$$

$$\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{d}{dx}\left(1 - \frac{2}{x^2+1}\right) = 0 - 2\frac{d}{dx}\{(x^2+1)^{-1}\}$$

$$= -2(-1)(x^2+1)^{-2}\frac{d}{dx}(x^2+1)$$

$$= 2(x^2+1)^{-2}(2x) = \frac{4x}{(x^2+1)^2}.$$

$$\frac{d}{dx}\{\log_e(\tan \tfrac{1}{2}x)\} = \frac{d}{dx}(\log_e z), \text{ where } z = \tan \tfrac{1}{2}x,$$

$$= \frac{d}{dz}(\log_e z) . \frac{dz}{dx}$$

$$= \frac{1}{z} . \tfrac{1}{2}\sec^2 \tfrac{1}{2}x$$

$$= \frac{\cos(x/2)}{\sin(x/2)} \times \frac{1}{2\cos^2(x/2)}$$

$$= \frac{1}{2\sin(x/2)\cos(x/2)} = \frac{1}{\sin x}$$

$$= \operatorname{cosec} x.$$

Example 7.—Find the derivatives with respect to x of

$$\text{(i) } x\sin^{-1}(2x-1), \qquad \text{(ii) } \frac{1}{x}\cosh 3x + 2x\sinh 3x,$$

$$\text{(iii) } \tanh^{-1} 2/(2+x), \qquad \text{(iv) } \cosh^{-1}\sqrt{(3-x)}.$$

(i) $$\frac{d}{dx}\{x\sin^{-1}(2x-1)\} = x\frac{d}{dx}\{\sin^{-1}(2x-1)\} + \sin^{-1}(2x-1)\frac{d}{dx}(x)$$

$$= x \cdot \frac{1}{\sqrt{\{1-(2x-1)^2\}}} \cdot \frac{d}{dx}(2x-1) + \sin^{-1}(2x-1)$$

$$= \frac{2x}{\sqrt{(4x-4x^2)}} + \sin^{-1}(2x-1)$$

$$= \frac{x}{\sqrt{(x-x^2)}} + \sin^{-1}(2x-1).$$

(ii) $$\frac{d}{dx}\left\{\frac{1}{x}\cosh 3x + 2x\sinh 3x\right\}$$

$$= \left\{\frac{1}{x}\frac{d}{dx}(\cosh 3x) + \cosh 3x\frac{d}{dx}\left(\frac{1}{x}\right)\right\} + \left\{2x\frac{d}{dx}(\sinh 3x) + \sinh 3x\frac{d}{dx}(2x)\right\}$$

$$= \frac{1}{x} \times 3\sinh 3x - \frac{1}{x^2}\cosh 3x + 2x(3\cosh 3x) + 2\sinh 3x$$

$$= \left(2+\frac{3}{x}\right)\sinh 3x + \left(6x - \frac{1}{x^2}\right)\cosh 3x.$$

(iii) $$\frac{d}{dx}\{\tanh^{-1} 2/(2+x)\} = \frac{d}{dx}(\tanh^{-1} z), \text{ where } z = \frac{2}{2+x},$$

$$= \frac{d}{dz}(\tanh^{-1} z)\frac{dz}{dx}$$

$$= \frac{1}{1-z^2} \cdot \frac{-2}{(2+x)^2}$$

$$= \frac{1}{1-\dfrac{4}{(2+x)^2}} \cdot \frac{-2}{(2+x)^2}$$

$$= \frac{-2}{(2+x)^2-4} = \frac{-2}{x^2+4x}.$$

(iv) $$\frac{d}{dx}\{\cosh^{-1}\sqrt{(3-x)}\} = \frac{d}{dx}(\cosh^{-1} z), \text{ where } z = (3-x)^{\frac{1}{2}},$$

$$= \frac{d}{dz}(\cosh^{-1} z) \cdot \frac{dz}{dx}$$

$$= \frac{\pm 1}{\sqrt{(z^2-1)}}(-\tfrac{1}{2})(3-x)^{-\frac{1}{2}}$$

$$= \frac{\pm 1}{2\sqrt{(3-x-1)}} \cdot \frac{1}{\sqrt{(3-x)}}$$

$$= \frac{\pm 1}{2\sqrt{\{(3-x)(2-x)\}}}.$$

8. Derivative of a continued product.

Let $y = uvw \ldots$, where $u, v, w \ldots$ are all functions of x, then it can be proved by an extension of the result for the differential coefficient of a product that

$$\frac{d}{dx}(uvw\ldots) = (vw\ldots)\frac{du}{dx} + (uw\ldots)\frac{dv}{dx} + (uv\ldots)\frac{dw}{dx} + \ldots .$$

The results for the differential coefficient of a continued product can also be obtained by taking logarithms to the base e of each side of the equation $y = uvw \ldots$, and then differentiating the resulting equation with respect to x.

N.B.—This latter method, known as *logarithmic differentiation*, cannot be used when one of the functions involved is already a logarithm.

Example 8.—Find (i) $\frac{d}{dx}(x^3e^{x^2}\sin 2x)$, (ii) x^{3x^2}.

(i) Let $\qquad y = x^3e^{x^2}\sin 2x.$

Taking logs to the base e,

$$\log_e y = 3\log_e x + x^2 + \log_e(\sin 2x). \quad \ldots \ldots \quad \text{(i)}$$

Differentiating (i) with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} + 2x + \frac{2\cos 2x}{\sin 2x};$$

$$\therefore \frac{dy}{dx} = x^3e^{x^2}\sin 2x\left[\frac{3}{x} + 2x + \frac{2\cos 2x}{\sin 2x}\right]$$

$$= x^2e^{x^2}[(3 + 2x^2)\sin 2x + 2x\cos 2x].$$

(ii) Although not a product, the logarithmic differentiation method is applicable in this case.

Let $\qquad y = x^{3x^2},$

$$\therefore \log_e y = 3x^2\log_e x. \quad \ldots \ldots \ldots \quad \text{(ii)}$$

Differentiating (ii) with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = 3x^2 \times \frac{1}{x} + 6x\log_e x$$

$$= 3x + 6x\log_e x,$$

$$\therefore \frac{dy}{dx} = x^{3x^2}(3x + 6x\log_e x),$$

$$\text{i.e. } \frac{d}{dx}(x^{3x^2}) = 3x^{3x^2+1}(1 + 2\log_e x).$$

9. Simplification before differentiation.

In the case of certain types of expressions, the differentiation i facilitated if the expression itself is simplified before differentiatior Notable cases are those in which logarithms occur, when use is mad of the theorems on logarithms; also the cases of fractions where th denominator is a surdic quantity, which are simplified by rationalizin the denominator.

Example 9.—Find $\frac{dy}{dt}$ where (*a*) $y = \log_e \sqrt{\frac{1+t^3}{1+t^2}}$, (*b*) $y = \frac{\sqrt{(t^2+a^2)} - t}{\sqrt{(t^2+a^2)} + t}$.

$$(a) \qquad y = \tfrac{1}{2}\log_e \frac{1+t^3}{1+t^2} = \tfrac{1}{2}\{\log_e(1+t^3) - \log_e(1+t^2)\}.$$

$$\therefore \frac{dy}{dt} = \frac{1}{2}\left\{\frac{3t^2}{1+t^3} - \frac{2t}{1+t^2}\right\} = \frac{t(t^3+3t-2)}{2(1+t^2)(1+t^3)}.$$

$$(b)\ y = \frac{(\sqrt{t^2+a^2} - t)^2}{(\sqrt{t^2+a^2}+t)(\sqrt{t^2+a^2}-t)} = \frac{2t^2+a^2-2t\sqrt{(t^2+a^2)}}{a^2}.$$

$$\therefore \frac{dy}{dt} = \frac{1}{a^2}\left[4t - 2\sqrt{(t^2+a^2)} - 2t \times \frac{t}{\sqrt{(t^2+a^2)}}\right]$$

$$= \frac{1}{a^2}\left[4t - \frac{2(t^2+a^2)+2t^2}{\sqrt{(t^2+a^2)}}\right]$$

$$= \frac{2}{a^2}\left[2t - \frac{(2t^2+a^2)}{\sqrt{(t^2+a^2)}}\right].$$

N.B.—In all cases the result must be simplified as far as possible.

10. Successive differentiation.

If $y = f(x)$, then the result of differentiating $\frac{dy}{dx}$ with respect to is known as the *second differential coefficient* (*or derivative*) *of* y *wit respect to* x, and is denoted by any one of the symbols

$$\frac{d}{dx}\left(\frac{dy}{dx}\right), \frac{d^2y}{dx^2}, \frac{d^2}{dx^2}\{f(x)\}, f''(x), D^2y, y_2.$$

Similarly, the result of differentiating $\frac{d^2y}{dx^2}$ with respect to x is know as the *third differential coefficient* (*or derivative*) *of* y *with respect to* and is denoted by any one of the symbols

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right), \frac{d^3y}{dx^3}, \frac{d^3}{dx^3}\{f(x)\}, f'''(x), D^3y, y_3.$$

Continuing in this manner, the result of differentiating y, n times successively with respect to x, is known as the nth *differential coefficient* (*or derivative*) *of* y *with respect to* x, and is denoted by any one of the symbols

$$\frac{d^n y}{dx^n}, \frac{d^n}{dx^n}[f(x)], f^{(n)}(x), D^n y, y_n.$$

11. Standard nth derivatives.

(*a*) e^{ax+b}. $\qquad \dfrac{d}{dx}(e^{ax+b}) = ae^{ax+b}$.

$$\frac{d^2}{dx^2}(e^{ax+b}) = \frac{d}{dx}(ae^{ax+b}) = a^2e^{ax+b}.$$

and continuing this process,

$$\frac{d^n}{dx^n}(e^{ax+b}) = a^n e^{ax+b}.$$

(*b*) a^{bx+c}. $\qquad \dfrac{d}{dx}(a^{bx+c}) = (b \log_e a)a^{bx+c}$.

$$\frac{d^2}{dx^2}(a^{bx+c}) = \frac{d}{dx}\{(b \log_e a)a^{bx+c}\}$$

$$= (b \log_e a)^2 a^{bx+c},$$

and in general, $\qquad \dfrac{d^n}{dx^n}(a^{bx+c}) = (b \log_e a)^n a^{bx+c}$.

(*c*) $\dfrac{1}{ax+b}$. $\qquad$ If $y = \dfrac{1}{ax+b} = (ax+b)^{-1}$,

using y_r for $\dfrac{d^r y}{dx^r}$, $\qquad y_1 = (-1)a(ax+b)^{-2}$

$$y_2 = (-1)(-2)a^2(ax+b)^{-3},$$

and in general,

$$y_n = (-1)(-2)\ldots(-n)a^n(ax+b)^{-(n+1)}$$

$$= \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}.$$

N.B.—To obtain the nth derivative with respect to x of the algebraic fraction $f(x)/\phi(x)$, it is advisable to express the given fraction in partial fractions, where possible, before differentiation.

Example 10.—Find the nth derivative with respect to x of $1/(x^2 - a^2)$.

Finding partial fractions by inspection,

$$\frac{1}{x^2 - a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right).$$

$$\therefore \frac{d^n}{dx^n}\left(\frac{1}{x^2 - a^2}\right) = \frac{1}{2a}\frac{d^n}{dx^n}\left\{\frac{1}{x-a} - \frac{1}{x+a}\right\}$$

$$= \frac{1}{2a}\left[\frac{(-1)^n n!}{(x-a)^{n+1}} - \frac{(-1)^n n!}{(x+a)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2a}\left[\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}}\right].$$

(*d*) $\log_e(ax + b)$. $\qquad \dfrac{d}{dx}\left\{\log_e(ax+b)\right\} = \dfrac{a}{ax+b}.$

$$\therefore \frac{d^n}{dx^n}\left\{\log_e(ax+b)\right\} = a\frac{d^{n-1}}{dx^{n-1}}\left[\frac{1}{ax+b}\right]$$

$$= \frac{a(-1)^{n-1}(n-1)!\,a^{n-1}}{(ax+b)^n}$$

$$= \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}.$$

(*e*) $\sin(ax + b)$ and $\cos(ax + b)$.

Let $\quad y = \sin(ax + b)$,

$$\therefore y_1 = a\cos(ax+b) = a\sin\left(ax + b + \frac{\pi}{2}\right),$$

$$y_2 = a^2\cos\left(ax + b + \frac{\pi}{2}\right) = a^2\sin\left(ax + b + \frac{2\pi}{2}\right),$$

$$y_3 = a^3\cos\left(ax + b + \frac{2\pi}{2}\right) = a^3\sin\left(ax + b + \frac{3\pi}{2}\right),$$

and in general, $\dfrac{d^n}{dx^n}\{\sin(ax+b)\} = a^n\sin\left(ax + b + \dfrac{n\pi}{2}\right).$

Similarly, $\dfrac{d^n}{dx^n}\{\cos(ax+b)\} = a^n\cos\left(ax + b + \dfrac{n\pi}{2}\right).$

(*f*) $e^{ax}\sin(bx + c)$ and $e^{ax}\cos(bx + c)$.

Let $\qquad y = e^{ax}\sin(bx + c)$,

$$\therefore y_1 = ae^{ax}\sin(bx+c) + be^{ax}\cos(bx+c)$$

$$= e^{ax}[a\sin(bx+c) + b\cos(bx+c)].$$

Let $\qquad a = r\cos\alpha,\quad$ where r is positive, $\quad . \quad . \quad . \quad$ (15)

$\qquad\qquad b = r\sin\alpha \quad$ and $\quad 0 \leqslant \alpha \leqslant 360°. \quad . \quad . \quad . \quad$ (16)

$(15)^2 + (16)^2$

$$a^2 + b^2 = r^2(\cos^2\alpha + \sin^2\alpha)$$
$$= r^2.$$
$$\therefore\ r = \sqrt{(a^2 + b^2)}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\left.\begin{aligned} \therefore\ \sin\alpha &= b/\sqrt{(a^2 + b^2)}, \\ \text{and} \qquad \cos\alpha &= a/\sqrt{(a^2 + b^2)}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Then $\quad y_1 = e^{ax}[r\sin(bx + c)\cos\alpha + r\cos(bx + c)\sin\alpha]$

$$= re^{ax}\sin(bx + c + \alpha).$$
$$y_2 = r[ae^{ax}\sin(bx + c + \alpha) + be^{ax}\cos(bx + c + \alpha)]$$
$$= re^{ax}[r\sin(bx + c + \alpha)\cos\alpha + r\cos(bx + c + \alpha)\sin\alpha]$$
$$= r^2e^{ax}\sin(bx + c + 2\alpha).$$

Proceeding in this manner it can be seen that

$$y_n = \frac{d^n}{dx^n}\{e^{ax}\sin(bx + c)\} = r^n e^{ax}\sin(bx + c + n\alpha).$$

Similarly, $\quad \dfrac{d^n}{dx^n}\{e^{ax}\cos(bx + c)\} = r^n e^{ax}\cos(bx + c + n\alpha).$

N.B.—$r = \sqrt{a^2 + b^2}$, and α is given by the two equations in (18), as both are needed to determine the quadrant in which α lies.

12. Leibnitz's Theorem.

This is used to find the nth derivative of a product, and states that, if $y = uv$, and u and v are functions of x, then

$$\begin{aligned} y_n = uv_n + {}^nC_1u_1v_{n-1} + {}^nC_2u_2v_{n-2} + \dots + {}^nC_{r-1}u_{r-1}v_{n-r+1} \\ + {}^nC_ru_rv_{n-r} + \dots + u_nv, \quad . \quad . \quad . \quad (19) \end{aligned}$$

where y_r, u_r, v_r are the rth differential coefficients of y, u, v respectively, with respect to x.

Proof by induction.

It will be assumed that the equation (19) holds true for the value n.

Differentiating each side of (19) with respect to x,

$$\begin{aligned} y_{n+1} &= (uv_{n+1} + u_1v_n) + {}^nC_1(u_1v_n + u_2v_{n-1}) + {}^nC_2(u_2v_{n-1} + u_3v_{n-2}) \\ &\qquad + \dots + {}^nC_{r-1}(u_{r-1}v_{n-r+2} + u_rv_{n-r+1}) \\ &\qquad + {}^nC_r(u_rv_{n-r+1} + u_{r+1}v_{n-r}) + \dots + (u_nv_1 + u_{n+1}v) \\ &= uv_{n+1} + ({}^nC_1 + 1)u_1v_n + ({}^nC_2 + {}^nC_1)u_2v_{n-1} + \dots \\ &\qquad + ({}^nC_r + {}^nC_{r-1})u_rv_{n-r+1} + \dots + u_{n+1}v. \end{aligned}$$

But ${}^nC_1 + 1 = n + 1 = {}^{n+1}C_1$, and, as shown in the proof of the binomial theorem, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$, for all positive integral values of n and r.

$$\therefore\ y_{n+1} = uv_{n+1} + {}^{n+1}C_1u_1v_n + {}^{n+1}C_2u_2v_{n-1} + \ldots$$
$$+ {}^{n+1}C_ru_rv_{n-r+1} + \ldots + u_{n+1}v. \quad . \quad (20)$$

But equation (20) is the same as if n were replaced by $(n + 1)$ in the equation (19). Hence, if Leibnitz's theorem be true for the value n, it is also true for the value $(n + 1)$ . . . . (A)

Now, by straightforward differentiation,

$$y_1 = uv_1 + u_1v.$$
$$y_2 = (uv_2 + u_1v_1) + (u_1v_1 + u_2v)$$
$$= uv_2 + 2u_1v_1 + u_2v$$
$$= uv_2 + {}^2C_1u_1v_1 + u_2v.$$

But these are the same results as obtained by using $n = 1$ and $n = 2$ respectively in equation (19), which shows that Leibnitz's theorem is true for $n = 1$ and $n = 2$. Hence by statement (A), it must be true for $n = 3$, and since it is true for $n = 3$, it must be true for $n = 4$, and so on for all positive integral values of n.

Thus, for all positive integral values of n, where $y = uv$,

$$\boldsymbol{y_n = uv_n + {}^nC_1u_1v_{n-1} + {}^nC_2u_2v_{n-2} + \ldots + {}^nC_ru_rv_{n-r} + \ldots + u_nv.}$$

Note.—The result is similar to the binomial expansion of $(u + v)^n$, but here suffixes are used instead of indices.

N.B.—In the application of this theorem, the u function should be a function of x (if possible) that vanishes after a finite number of differentiations.

Example 11.—Find the nth derivative of $x^3 \sin ax$ with respect to x.

Let $u = x^3$, $\therefore u_1 = 3x^2$, $u_2 = 6x$, $u_3 = 6$, $u_4 = u_5 = \ldots = 0$.

Let $v = \sin ax$, $\therefore v_n = a^n \sin\left(ax + \frac{n\pi}{2}\right)$, $v_{n-1} = a^{n-1} \sin\left(ax + \frac{n-1}{2}\pi\right)$,

$$v_{n-2} = a^{n-2} \sin\left(ax + \frac{n-2}{2}\pi\right),\ v_{n-3} = a^{n-3} \sin\left(ax + \frac{n-3}{2}\pi\right).$$

If $y = uv$, using Leibnitz's theorem,

$$y_n = a^nx^3 \sin\left(ax + \frac{n\pi}{2}\right) + 3\,.\,{}^nC_1a^{n-1}x^2 \sin\left(ax + \frac{n-1}{2}\pi\right)$$
$$+ 6\,.\,{}^nC_2a^{n-2}x \sin\left(ax + \frac{n-2}{2}\pi\right) + 6\,.\,{}^nC_3a^{n-3} \sin\left(ax + \frac{n-3}{2}\pi\right).$$

Example 12 (L.U.).—If $y = \dfrac{1}{\sqrt{(1-x^2)}} \sin^{-1} x$, prove that $(1-x^2)\dfrac{dy}{dx} = xy + 1$.

By applying Leibnitz's theorem, show that

$$(1-x^2)y_{n+1} - (2n+1)xy_n - n^2y_{n-1} = 0, \text{ where } y_r = \frac{d^ry}{dx^r}.$$

Hence find the value of y_{n+1} when $x = 0$.

N.B.—In this type of problem it is advisable to clear fractions at any stage before differentiating, as it is more convenient to differentiate a product rather than a fraction.

$$y = \frac{1}{\sqrt{(1-x^2)}} \sin^{-1} x, \quad \therefore y\sqrt{1-x^2} = \sin^{-1} x. \quad . \quad . \quad . \quad \text{(i)}$$

Differentiating (i) with respect to x,

$$\sqrt{1-x^2}\frac{dy}{dx} - \frac{x}{\sqrt{(1-x^2)}} y = \frac{1}{\sqrt{(1-x^2)}},$$

i.e. $$(1-x^2)\frac{dy}{dx} - xy = 1,$$

i.e. $$(1-x^2)\frac{dy}{dx} = xy + 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

The nth derivative of the L.H.S. of (ii) will equal the nth derivative of the R.H.S. of (ii), and applying Leibnitz's theorem, the nth derivative of $(1-x^2)y_1$ with respect to x is

$$(1-x^2)y_{n+1} + {}^nC_1(-2x)y_n + {}^nC_2(-2)y_{n-1}$$
$$= (1-x^2)y_{n+1} - 2nxy_n - n(n-1)y_{n-1}. \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

The nth derivative of $(xy + 1)$ with respect to x is

$$xy_n + ny_{n-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Equating (iii) and (iv),

$$(1-x^2)y_{n+1} - 2nxy_n - n(n-1)y_{n-1} = xy_n + ny_{n-1},$$

i.e. $$(1-x^2)y_{n+1} - (2n+1)xy_n - n^2y_{n-1} = 0. \quad . \quad . \quad . \quad . \quad \text{(v)}$$

Using $x = 0$ in (v), $$y_{n+1} = n^2y_{n-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

In (vi),

when $n = 1$, $\quad y_2 = y_0 = y = 0$. [$y = 0$ when $x = 0$ from (i).]

When $n = 3$, $\quad y_4 = 3^2y_2 = 0$, etc.

$\therefore$ when n is odd, $y_{n+1} = 0$.

In (vi),

when $n = 2$, $\quad y_3 = 2^2y_1 = 2^2$. [When $x = 0$, $y_1 = 1$ from (ii).]

When $n = 4$, $\quad y_5 = 4^2y_3 = 2^2 . 4^2$.

When $n = 6$, $\quad y_7 = 6^2y_5 = 2^2 . 4^2 . 6^2$.

. . . .

When $n = 2r$, $\quad y_{2r+1} = 2^2 . 4^2 . 6^2 \ldots (2r)^2$

$$= (2^2)^r(r!)^2 = 2^{2r}(r!)^2.$$

$$\therefore y_{n+1} = 2^n\{(n/2)!\}^2, \text{ since } r = n/2.$$

Therefore, when n is odd, $y_{n+1} = 0$, and when n is even, $y_{n+1} = 2^n\{(n/2)!\}^2$.

Example 13 (L.U.).—If $y = e^{\alpha \sin^{-1} x}$, show that

(*a*) $(1 - x^2)y_2 - xy_1 - \alpha^2 y = 0$,

(*b*) $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + \alpha^2)y_n = 0$.

Hence, or otherwise, expand y into a series in ascending powers of x, giving explicitly the coefficients of x^{2r} and x^{2r+1}.

$$y = e^{\alpha \sin^{-1} x}. \qquad \text{(i)}$$

$$\therefore \frac{dy}{dx} = \left\{\frac{d}{dx}(\alpha \sin^{-1} x)\right\} e^{\alpha \sin^{-1} x}$$

$$= \frac{\alpha}{\sqrt{(1 - x^2)}} e^{\alpha \sin^{-1} x}.$$

$$\therefore (1 - x^2)^{\frac{1}{2}} y_1 = \alpha y. \qquad \text{(ii)}$$

Differentiating (ii) with respect to x,

$$(1 - x^2)^{\frac{1}{2}} y_2 - \frac{x}{(1 - x^2)^{\frac{1}{2}}} y_1 = \alpha \frac{dy}{dx}$$

$$= \frac{\alpha^2}{(1 - x^2)^{\frac{1}{2}}} y.$$

$$\therefore (1 - x^2)y_2 - xy_1 = \alpha^2 y,$$

$$(1 - x^2)y_2 - xy_1 - \alpha^2 y = 0. \qquad \text{(iii)}$$

Using Leibnitz's theorem,

$$\frac{d^n}{dx^n}\{(1 - x^2)y_2\} = (1 - x^2)y_{n+2} - 2nxy_{n+1} + \frac{n(n - 1)}{2!}(-2)y_n$$

$$= (1 - x^2)y_{n+2} - 2nxy_{n+1} - n(n - 1)y_n.$$

$$\frac{d^n}{dx^n}(xy_1) = xy_{n+1} + ny_n.$$

$$\frac{d^n}{dx^n}(\alpha^2 y) = \alpha^2 y_n.$$

Thus, differentiating equation (iii) n times successively with respect to x,

$$\{(1 - x^2)y_{n+2} - 2nxy_{n+1} - n(n - 1)y_n\} - (xy_{n+1} + ny_n) - \alpha^2 y_n = 0,$$

i.e.
$$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + \alpha^2)y_n = 0. \qquad \text{(iv)}$$

Let $y = A_0 + A_1 x + \ldots + A_n x^n + A_{n+1}x^{n+1} + A_{n+2}x^{n+2} + \ldots$

$$y_n = n!\,A_n + \frac{(n + 1)!}{1!} A_{n+1}x + \frac{(n + 2)!}{2!} A_{n+2}x^2 + \ldots$$

$$y_{n+1} = (n + 1)!\,A_{n+1} + \frac{(n + 2)!}{1!} A_{n+2}x + \ldots$$

$$y_{n+2} = (n + 2)!\,A_{n+2} + \ldots$$

Using these in (iv),

$$(1-x^2)\{(n+2)!\,A_{n+2}+\ldots\}-(2n+1)x\{(n+1)!\,A_{n+1}+\frac{(n+2)!}{1!}A_{n+2}x+\ldots\}$$
$$-(n^2+\alpha^2)\{n!\,A_n+\frac{(n+1)!}{1!}A_{n+1}x+\frac{(n+2)!}{2!}A_{n+2}x^2+\ldots\}=0.$$

This is an identity, being true for all values of x, and therefore, equating coefficients of unity,

$$(n+2)!\,A_{n+2}=(n^2+\alpha^2)A_n n!,$$
$$\therefore\ A_{n+2}=\frac{n^2+\alpha^2}{(n+1)(n+2)}A_n. \quad \ldots\ldots \quad \text{(v)}$$

From (i), when $x=0$, $y=1$, $\therefore\ A_0=1$ (using the series with $x=0$).

From (ii), when $x=0$, $y_1=\alpha y=\alpha$, $\therefore\ A_1=\alpha(y_1=A_1+2A_2x+\ldots)$.

From the sequence (v),

taking $n=0$, $\quad A_2=\frac{\alpha^2}{2}A_0=\frac{\alpha^2}{2};$

$n=2$, $\quad A_4=\frac{\alpha^2+2^2}{3\times 4}A_2=\frac{\alpha^2(\alpha^2+2^2)}{2\times 3\times 4}=\frac{\alpha^2(\alpha^2+2^2)}{4!};$

$n=4$, $\quad A_6=\frac{\alpha^2+4^2}{5\times 6}A_4=\frac{\alpha^2(\alpha^2+2^2)(\alpha^2+4^2)}{6!};$

and, in general, $\quad A_{2r}=\frac{\alpha^2(\alpha^2+2^2)(\alpha^2+4^2)\ldots\{\alpha^2+(2r-2)^2\}}{(2r)!}.$

Also in (v),

taking $n=1$, $\quad A_3=\frac{\alpha^2+1^2}{2\times 3}\cdot A_1=\alpha\cdot\frac{\alpha^2+1^2}{2\times 3}=\frac{\alpha(\alpha^2+1^2)}{3!};$

$n=3$, $\quad A_5=\frac{\alpha^2+3^2}{4\times 5}\cdot A_3=\frac{\alpha(\alpha^2+1)(\alpha^2+3^2)}{5!}$, etc.;

and, in general, $\quad A_{2r+1}=\frac{\alpha(\alpha^2+1^2)(\alpha^2+3^2)\ldots\{\alpha^2+(2r-1)^2\}}{(2r+1)!}.$

13. Parametric equation of a curve.

If a curve be given in the form $x=f(t)$, $y=\phi(t)$, this is known as the *parametric equation* of the curve and t is known as the *parameter*.

To obtain the Cartesian equation (equation in x and y) of the curve from the parametric equation, it is necessary to eliminate the parameter t between the values of x and y. The table for drawing the curve can be obtained by taking various values for t and tabulating the corresponding values of x and y.

Thus, the equations $x=a\cos nt$, $y=a\sin nt$, produce, on elimination of t by squaring and adding, the Cartesian equation $x^2+y^2=a^2$, which is the equation of a circle of radius a and centre the origin.

Also, the parametric equation $x = at^2$, $y = 2at$, produces the Cartesian equation $y^2 = 4ax$ on elimination of t, which is the equation of a parabola, as will be shown later.

The fluxional notation is often adopted when dealing with parametric equations,

i.e. $\dot{x}$ is used for $\dfrac{dx}{dt}$, $\dot{y}$ for $\dfrac{dy}{dt}$, $\ddot{x}$ for $\dfrac{d^2x}{dt^2}$, $\ddot{y}$ for $\dfrac{d^2y}{dt^2}$, and so on.

14. Theorem.—To find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$, when the equation of the curve is given in the parametric form $x = f(t)$, $y = \phi(t)$.

Using the function of a function theorem,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{dy}{dt}\bigg/\frac{dx}{dt} = \frac{\dot{y}}{\dot{x}}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\dot{y}}{\dot{x}}\right) = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) \cdot \frac{dt}{dx}$$

$$= \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x})^2} \times \frac{1}{dx/dt}$$

$$= \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x})^3}.$$

Example 14.—If $x = a(\theta - \sin\theta)$, $y = a(\theta + \sin\theta)$, find the values of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ when $\theta = \dfrac{\pi}{2}$.

Using $\dot{x} = \dfrac{dx}{d\theta}$, $\dot{y} = \dfrac{dy}{d\theta}$, etc.,

$$\dot{x} = a(1 - \cos\theta), \quad \ddot{x} = a\sin\theta,$$

$$\dot{y} = a(1 + \cos\theta), \quad \ddot{y} = -a\sin\theta.$$

When $\theta = \pi/2$, $\quad \dot{x} = \dot{y} = a, \quad \ddot{x} = a, \quad \ddot{y} = -a.$

$$\therefore \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{a}{a} = 1.$$

$$\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^3} = \frac{-a^2 - a^2}{a^3} = \frac{-2}{a}.$$

EXAMPLES ON CHAPTER IV

All questions except the first are taken from London University papers.

1. Differentiate the following functions with respect to the variable contained in each:

(a) $\dfrac{[\text{illegible}]}{(3x-2)^5}$, (b) $\sqrt{1-2t}$, (c) $1/(3s^2)^{\frac{1}{3}} + s^{-0\cdot 2}$,

(d) $\dfrac{1}{3e^{2x}}$, (e) $(y+1)^3 e^{-y+1}$, (f) $3^{2z+3} - 5\log_e(1-z)$,

(g) $\dfrac{\sqrt{(x^2+4)} - x}{\sqrt{(x^2+4)} + x}$, (h) $\log_e \dfrac{\sqrt{(1-2z)}}{3z}$, (i) $\sin 3\theta^\circ$,

(j) $3\cos^2 2\theta$, (k) $\log_e(\tan \frac{1}{2}\phi)$, (l) $t \sin^{-1}(2t-1)$,

(m) $\dfrac{p}{(p+1)(p+2)}$, (n) $y/\cos^{-1}(3y)$, (o) $\sin 2x \cot x$,

(p) $\sec^2\phi + \tan^3\phi$, (q) $\sin^{-1}\dfrac{2y}{1+y^2}$, (r) $\sec^{-1} z/2$,

(s) $\sinh^{-1}(1-s)^{\frac{1}{2}}$, (t) $3\tanh^{-1} y/4$, (u) $\log_e(x^3 \cosh^{-1} x)$,

(v) $\sqrt{\operatorname{sech} 3\theta}$, (w) $z^2 e^{3z} \operatorname{cosech} z$, (x) $\log_e(1-2x+2x^2)^{\frac{3}{2}}$,

(y) $\operatorname{sech}^2\theta \tanh 2\theta$, (z) $\sinh^{-1}(\cot x/2)$.

2. (i) Differentiate with respect to x the functions $\log_e\{(1+x^{\frac{1}{2}})/(1-x^{\frac{1}{2}})\}$ and $\operatorname{sech}^{-1}x$.

(ii) If $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$, find the value of $\dfrac{d^2y}{dx^2}$, when $t = \pi/2$.

3. Find from first principles the differential coefficient of $\cot x$ with respect to x.

If $(p+qx)e^{y/x} = x$, where p and q are constants, prove that

$$x^3 \frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2.$$

4. From first principles, differentiate $\sec x$ with respect to x.

If $y = \dfrac{(3+x)(2-x)^2}{(4-3x)^3}$, find the values of x when $\dfrac{dy}{dx} = 0$.

5. Differentiate with respect to x:

$$\text{(i) } \sin^{-1}\left\{\frac{x^2}{(x^4+a^4)^{\frac{1}{2}}}\right\}, \quad \text{(ii) } \log_e\left\{e^x\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}.$$

If $x = \sin t$, $y = \sin pt$, prove that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0.$$

6. (i) Find the limits of (*a*) $\log_e(1 + ax) - 2\log_e x + \log_e(a + x)$,
(*b*) $\{(x^2 + ax + b)^{\frac{1}{2}} - x\}$, as x tends to infinity.

(ii) Differentiate $\log_e\{(a - x)/(a + x)\} + 2\tanh^{-1} x/a$, with respect to x.

7. Find from first principles the derivative of $x \cos x$ with respect to x.

Express in their simplest forms the derivatives with respect to x of

$$\text{(i)}\ \tan^{-1}\left\{\frac{\sqrt{x}(3 - x)}{1 - 3x}\right\}, \qquad \text{(ii)}\ x^a \log_e x.$$

Find the limit as $x \to 0$ of $\dfrac{\sin^2 x - x^2 \cos x}{x^3 \tan x}$.

8. (i) Find $\underset{a \to 0}{\text{Lt}} \dfrac{ab(\cosh ab - \cosh ax)}{ab \cosh ab - \sinh ab}$.

(ii) Differentiate $\log_e\{\tan(\pi/4 + x/2)\}$ and $2\tanh^{-1}(\tan \frac{1}{2}x)$. What deduction can be made from these results?

9. Prove that $\dfrac{dx}{dy} = 1\Big/\dfrac{dy}{dx}$, and find a formula for $\dfrac{d^2x}{dy^2}$ in terms of $\dfrac{d^2y}{dx^2}$ and $\dfrac{dy}{dx}$.

Verify the formula by applying it to the two cases (i) $y = x^2$, (ii) $y = e^x$.

10. Define $\sinh x$, $\cosh x$, $\tanh x$, and $\operatorname{sech} x$.

Prove from your definitions that

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x.$$

If $y = \tanh x$, show that $\dfrac{d^3y}{dx^3} + 2\left\{\left(\dfrac{dy}{dx}\right)^2 + y\dfrac{d^2y}{dx^2}\right\} = 0$.

11. If $y = e^{ax}\cos bx$, show that $\dfrac{d^ny}{dx^n} = r^n e^{ax}\cos(bx + n\phi)$, where $r^2 = a^2 + b^2$ and $\tan\phi = b/a$.

By use of Leibnitz's theorem, or otherwise, show that

$$r^n \cos n\phi = a^n - {}^nC_2 a^{n-2}b^2 + {}^nC_4 a^{n-4}b^4 - {}^nC_6 a^{n-6}b^6 + \dots .$$

12. (i) Show by induction, or otherwise, that the nth derivative of $e^x \cos x$ with respect to x is $2^{(\frac{1}{2})n} e^x \cos(x + \frac{1}{4}n\pi)$.

(ii) If $x = a\sin t - b\sin\left(\dfrac{a}{b}t\right)$, $y = a\cos t - b\cos\left(\dfrac{a}{b}t\right)$, where a and b are constants, find expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t.

13. If $y = (x + \sqrt{x^2 + 1})^p$, prove that

$$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - p^2y = 0.$$

Differentiate this equation n times by Leibnitz's theorem, and hence, or otherwise, show that the expansion of y in ascending powers of x is

$$1 + px + \tfrac{1}{2}p^2x^2 + \dots + \frac{p(p^2 - 1^2)(p^2 - 3^2)\dots(p^2 - \overline{2n - 1}^2)}{(2n + 1)!}x^{2n+1}$$
$$+ \frac{p^2(p^2 - 2^2)(p^2 - 4^2)\dots(p^2 - 4n^2)}{(2n + 2)!}x^{2n+2} + \dots .$$

14. (i) If $y = \dfrac{x}{1-x^2}$, find $\dfrac{d^ny}{dx^n}$ in terms of x.

(ii) If $y = \dfrac{x}{1+x^2}$, prove that $\dfrac{d^ny}{dx^n} = (-1)^n n! \cos(n+1)\theta \sin^{n+1}\theta$, where $x = \cot\theta$.

15. Prove that $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$, u and v being functions of x.

If n be a positive integer less than p, prove, by induction, or otherwise, that

$$\left(\frac{d}{dx}\right)^n (x^p \log_e x)$$
$$= x^{p-n} . p(p-1) \ldots (p-n+1)\left\{\log_e x + \frac{1}{p} + \frac{1}{p-1} + \ldots + \frac{1}{p-n+1}\right\}.$$

16. If $y = [\log_e\{x + (a^2 + x^2)^{\frac{1}{2}}\}]^2$, show that

$$(a^2 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2.$$

Differentiate this equation n times and deduce, or find by any other means, the expansion for y in terms of positive integral powers of x, giving the general term.

17. If $y = (\sin^{-1} x)^2$, show that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 2$.

Apply Leibnitz's theorem to this equation to find a relation between y_n, y_{n+1}, y_{n+2}, where $y_r = \dfrac{d^ry}{dx^r}$.

Hence, show that, if $(\sin^{-1} x)^2$ be expanded in a series of ascending powers of x, the coefficient of $x^{2n}(n \geqslant 1)$ is $2^{2n-1}\{(n-1)!\}^2/(2n)!$.

18. If $y = (x^2-1)^n$, prove that $(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$.

If $P = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\{(x^2-1)^n\}$, show that P satisfies the equation

$$(1-x^2)y_2 - 2xy_1 + n(n+1)y = 0, \text{ where } y_r = \frac{d^ry}{dx^r}.$$

19. If $y = (1-x)^{-\alpha}e^{-\alpha x}$, show that

$$\text{(i) } (1-x)\frac{dy}{dx} = \alpha xy,$$

$$\text{(ii) } (1-x)y_{n+1} - (n+\alpha x)y_n - n\alpha y_{n-1} = 0.$$

Expand y in ascending powers of x as far as the term in x^5.

20. If $y = (1-x^2)^{\frac{1}{2}} \sin^{-1} x$, prove that

(i) $(1-x^2)\dfrac{dy}{dx} + xy = 1 - x^2$,

(ii) $(1-x^2)\dfrac{d^{n+1}y}{dx^{n+1}} - (2n-1)x\dfrac{d^ny}{dx^n} - n(n-2)\dfrac{d^{n-1}y}{dx^{n-1}} = 0 \ldots (n > 2)$.

Hence, or otherwise, expand y in a series of ascending powers of x, as far as the term containing x^7.

21. If $y = \dfrac{\sin^{-1} x}{(1 - x^2)^{\frac{1}{2}}}$, show that $(1 - x^2)\dfrac{dy}{dx} - xy = 1$.

By differentiating this equation n times, or by any other method, show that

$$\frac{(\sin^{-1} x)}{(1 - x^2)^{\frac{1}{2}}} = x + \sum_{n=2}^{\infty} \frac{2 . 4 \ldots (2n - 2)}{3 . 5 \ldots (2n - 1)} x^{2n-1}.$$

22. If $y = x^m \log_e x$, show that $x\dfrac{dy}{dx} = my + x^m$.

Differentiate this equation n times, where $n > m$.

23. If $y = \left(\dfrac{1 + x}{1 - x}\right)^r$, prove that $(1 - x^2)\dfrac{dy}{dx} = 2ry$.

Differentiate this n times by Leibnitz's theorem, and obtain the relation between any three consecutive coefficients in the expansion of y as a series in ascending powers of x.

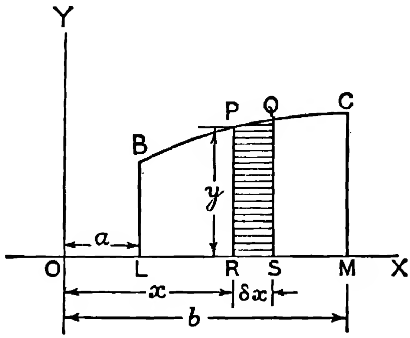
CHAPTER V

Integration

1. Integration is the converse of differentiation. Thus, if $\frac{d}{dx}[f(x)] = \phi(x)$, then $f(x)$ is said to be "the integral of $\phi(x)$ with respect to x", and this is written symbolically as $\int\phi(x)dx = f(x)$, where $\int$ is an elongated S (for "summation"), and represents "the integral of", and the dx means "with respect to x". The quantity $\phi(x)$ is known as the *integrand*.

It has been proved in Chap. IV that $\frac{d}{dx}[f(x) + C] = \frac{d}{dx}[f(x)]$, where C is *any* constant.

Hence $\int\phi(x)dx = f(x) + C$, where C is an arbitrary constant, if $\frac{d}{dx}\{f(x)\} = \phi(x)$.



The result $f(x)$ obtained for the integration is known as the *indefinite integral*, and the result $f(x) + C$, where C is an arbitrary constant, is known as the *general integral*.

The arbitrary constant C is usually called the *constant of integration* and will be thus used throughout this chapter.

2. Integration considered as an area.

Let $y = \phi(x)$ be represented by the curve shown in the diagram and let B and C be points on the curve whose abscissæ are a and b

respectively, with P and Q two adjacent points on the curve whose co-ordinates are (x, y), $(x + \delta x, y + \delta y)$ respectively. BL, PR, QS, CM are the ordinates of B, P, Q, C respectively, and δA is a small increase in the area A under the curve (i.e. between the curve, the x-axis, and the two ordinates) due to a small increase δx in x.

Then δA = shaded area PQSR,

i.e. $$\delta A \approx \delta x\left(y + \frac{\delta y}{2}\right) \quad \text{(PQSR approximately a trapezium),}$$

i.e. $$\frac{\delta A}{\delta x} \approx y + \frac{\delta y}{2}.$$

Proceeding to the limit as $\delta x \to 0$, and $\therefore\ \delta y \to 0$, $\dfrac{dA}{dx} = y$.

$$\therefore\ A = \int y\,dx = \int \phi(x)\,dx = f(x) + C, \quad . \quad . \quad (1)$$

where $\phi(x) = \dfrac{d}{dx}\{f(x)\}$, and C is an arbitrary constant. If the area A is to be considered as the area under the curve BC, then the area A will be zero at B, i.e. $A = 0$ when $x = a$.

$$\therefore\ \text{from (1), } 0 = f(a) + C; \quad \therefore\ C = -f(a);$$
$$\therefore\ A = f(x) - f(a).$$

Now the area A is required when $x = b$,

hence $$A = f(b) - f(a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From the result (1) it can be seen that an integral corresponds to a certain area, and from the result (2) it can be seen that, if the area be taken under the curve $y = \phi(x)$ between $x = a$ and $x = b$, its value is $f(b) - f(a)$, where $\dfrac{d}{dx}\{f(x)\} = \phi(x)$, i.e. $f(x)$ is the indefinite integral of $\phi(x)$ with respect to x.

In this latter case the integral representing the area is usually written $\int_a^b \phi(x)\,dx$ to show the limits $x = a$ and $x = b$ for the area, and the a and b in the integral are known as the *lower* and *upper limits* respectively, whilst the integral itself is known as a *definite integral*.

Thus $$A = \int_a^b \phi(x)\,dx = \Big[f(x)\Big]_a^b = f(b) - f(a).$$

Note.—The step $\Big[f(x)\Big]_a^b$ is inserted so as to simplify the setting out and eradicate certain errors which are liable to occur otherwise.

3. The following theorems on integration follow from the theorems

on differentiation and the fact that integration is the converse of differentiation.

Theorem 1.—$\int a\,dx = ax + C$, **where a is any constant.**

Theorem 2.—$\int a\varphi(x)\,dx = a\int\varphi(x)\,dx$, **where a is any constant.**

Theorem 3.—The integral of a sum is equal to the sum of the separate integrals.

4. Standard integrals.

The table on pp. 126–7 gives standard indefinite integrals obtained by considering integration as the converse of differentiation.

N.B.— $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\int\left(\frac{1}{a+x} + \frac{1}{a-x}\right)dx$ (partial fractions by inspection)

$$= \frac{1}{2a}\left[\log_e(a+x) - \log_e(a-x)\right]$$

$$= \frac{1}{2a}\log_e\frac{a+x}{a-x} \qquad (x^2 < a^2).$$

Similarly $$\int\frac{dx}{x^2 - a^2} = \frac{1}{2a}\log_e\frac{x-a}{x+a} \qquad (x^2 > a^2).$$

These results, which are equivalent to $\frac{1}{a}\tanh^{-1} x/a$ and $-\frac{1}{a}\coth^{-1} x/a$ respectively, as shown in Chap. II, are more convenient for calculation purposes and are generally used in preference to the values $\frac{1}{a}\tanh^{-1} x/a$ and $-\frac{1}{a}\coth^{-1} x/a$.

Also two results, obtained by different methods, may appear to differ, but may only differ by a constant which is permissible in integration.

5. Further standard integrals.

1. To find $\int\frac{f'(x)}{f(x)}dx$, and $\int\tan x\,dx$, $\int\cot x\,dx$.

Since $$\frac{d}{dx}(\log_e z) = \frac{d}{dz}(\log_e z)\cdot\frac{dz}{dx} = \frac{1}{z}\frac{dz}{dx}$$

$$= \frac{f'(x)}{f(x)}, \text{ where } z = f(x),$$

it follows that $$\int\frac{f'(x)}{f(x)}dx = \log_e\{f(x)\} + C.$$

Function of x	Integral with respect to x	General function of x	Integral with respect to x
x^n	$x^{n+1}/(n+1) \quad (n \neq -1)$	$(ax+b)^n$	$\dfrac{(ax+b)^{n+1}}{a(n+1)} \quad (n \neq -1)$
$\dfrac{1}{x} = x^{-1}$	$\log_e x$	$\dfrac{1}{ax+b} = (ax+b)^{-1}$	$\dfrac{1}{a} \log_e (ax+b)$
e^x	e^x	e^{ax+b}	$\dfrac{1}{a} e^{ax+b}$
a^x	$\dfrac{a^x}{\log_e a}$	a^{bx+c}	$\dfrac{1}{b \log_e a} \cdot a^{bx+c}$
$\cos x$	$\sin x$	$\cos(ax+b)$	$\dfrac{1}{a} \sin(ax+b)$
$\sin x$	$-\cos x$	$\sin(ax+b)$	$-\dfrac{1}{a} \cos(ax+b)$
$\sec^2 x = \dfrac{1}{\cos^2 x}$	$\tan x$	$\sec^2(ax+b) = \dfrac{1}{\cos^2(ax+b)}$	$\dfrac{1}{a} \tan(ax+b)$
$\sec x \tan x = \dfrac{\sin x}{\cos^2 x}$	$\sec x$	$\sec(ax+b) \tan(ax+b)$	$\dfrac{1}{a} \sec(ax+b)$
$\operatorname{cosec} x \cot x = \dfrac{\cos x}{\sin^2 x}$	$-\operatorname{cosec} x$	$\operatorname{cosec}(ax+b) \cot(ax+b)$	$-\dfrac{1}{a} \operatorname{cosec}(ax+b)$
$\operatorname{cosec}^2 x = 1/\sin^2 x$	$-\cot x$	$\operatorname{cosec}^2(ax+b)$	$-\dfrac{1}{a} \cot(ax+b)$
$\dfrac{1}{\sqrt{(1-x^2)}} \ (x^2 < 1)$	$\sin^{-1} x$ or $-\cos^{-1} x$	$\dfrac{1}{\sqrt{(a^2-x^2)}} \ (x^2 < a^2)$	$\sin^{-1} \dfrac{x}{a}$ or $-\cos^{-1} \dfrac{x}{a}$
$\dfrac{1}{x\sqrt{(x^2-1)}} \ (x^2 > 1)$	$\sec^{-1} x$ or $-\operatorname{cosec}^{-1} x$	$\dfrac{1}{x\sqrt{(x^2-a^2)}} \ (x^2 > a^2)$	$\dfrac{1}{a} \sec^{-1} \dfrac{x}{a}$ or $-\dfrac{1}{a} \operatorname{cosec}^{-1} \dfrac{x}{a}$
$\dfrac{1}{1+x^2}$	$\tan^{-1} x$ or $-\cot^{-1} x$	$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a} \tan^{-1} \dfrac{x}{a}$ or $-\dfrac{1}{a} \cot^{-1} \dfrac{x}{a}$

$\cosh x$	$\sinh x$	$\cosh(ax+b)$	$\frac{1}{a}\sinh(ax+b)$
$\sinh x$	$\cosh x$	$\sinh(ax+b)$	$\frac{1}{a}\cosh(ax+b)$
$\operatorname{sech}^2 x = \frac{1}{\cosh^2 x}$	$\tanh x$	$\operatorname{sech}^2(ax+b)$	$\frac{1}{a}\tanh(ax+b)$
$\operatorname{sech} x \tanh x = \sinh x/\cosh^2 x$	$-\operatorname{sech} x$	$\operatorname{sech}(ax+b)\tanh(ax+b)$	$-\frac{1}{a}\operatorname{sech}(ax+b)$
$\operatorname{cosech} x \coth x = \cosh x/\sinh^2 x$	$-\operatorname{cosech} x$	$\operatorname{cosech}(ax+b)\coth(ax+b)$	$-\frac{1}{a}\operatorname{cosech}(ax+b)$
$\operatorname{cosech}^2 x = 1/\sinh^2 x$	$-\coth x$	$\operatorname{cosech}^2(ax+b)$	$-\frac{1}{a}\coth(ax+b)$
$\frac{1}{\sqrt{(x^2+1)}}$	$\sinh^{-1} x$	$\frac{1}{\sqrt{(x^2+a^2)}}$	$\sinh^{-1}\frac{x}{a}$
$\frac{1}{\sqrt{(x^2-1)}}\ (x^2>1)$	$\pm\cosh^{-1} x$	$\frac{1}{\sqrt{(x^2-a^2)}}\ (a^2<x^2)$	$\pm\cosh^{-1}\frac{x}{a}$
$\frac{1}{1-x^2}\ (x^2<1)$	$\tanh^{-1} x$	$\frac{1}{a^2-x^2}\ (x^2<a^2)$	$\frac{1}{a}\tanh^{-1}\frac{x}{a}$
$\frac{1}{x\sqrt{(1-x^2)}}\ (x^2<1)$	$\pm\operatorname{sech}^{-1} x$	$\frac{1}{x\sqrt{(a^2-x^2)}}\ (x^2<a^2)$	$\pm\frac{1}{a}\operatorname{sech}^{-1}\frac{x}{a}$
$\frac{1}{x\sqrt{(x^2+1)}}$	$\pm\operatorname{cosech}^{-1} x$	$\frac{1}{x\sqrt{(x^2+a^2)}}$	$\pm\frac{1}{a}\operatorname{cosech}^{-1}\frac{x}{a}$
$\frac{1}{x^2-1}\ (x^2>1)$	$-\coth^{-1} x$	$\frac{1}{x^2-a^2}\ (x^2>a^2)$	$-\frac{1}{a}\coth^{-1}\frac{x}{a}$

From this result it can be seen that

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{\frac{d}{dx}(\cos x)}{\cos x}\,dx$$

$$= -\log_e(\cos x) + C.$$

$$\therefore \int \tan x\,dx = \log_e(\sec x) + C.$$

Similarly $$\int \cot x\,dx = \log_e(\sin x) + C.$$

2. To find $\int \frac{f'(x)}{\sqrt{\{f(x)\}}}\,dx.$

Since $$\frac{d}{dx}(\sqrt{z}) = \frac{d}{dz}(z^{\frac{1}{2}})\frac{dz}{dx} = \tfrac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dx}$$

$$= \frac{f'(x)}{2\sqrt{\{f(x)\}}}, \text{ where } z = f(x),$$

it follows that $$\int \frac{f'(x)}{\sqrt{\{f(x)\}}}\,dx = 2\sqrt{\{f(x)\}} + C.$$

N.B.—When finding a given integral, it must first be considered if it is one of the standard integrals already given, in which case the result can be obtained by inspection. If it is not one of these standard functions, it must be determined by one of the later methods. Indefinite integrals are only given when specifically requested.

***Example* 1.**—Find the following indefinite integrals.

(i) $\int \coth 3x\,dx$, (ii) $\int \frac{\sin 2\theta + 3\cos 3\theta}{\cos 2\theta - 2\sin 3\theta}\,d\theta$,

(iii) $\int \frac{x^2 + 2x}{\sqrt{(x^3 + 3x^2 + 2)}}\,dx$, (iv) $\int \frac{1 - 2t}{\sqrt{(3 + t - t^2)}}\,dt$.

(i) $$\int \coth 3x\,dx = \int \frac{\cosh 3x}{\sinh 3x}\,dx = \int \frac{\frac{1}{3}\frac{d}{dx}(\sinh 3x)}{\sinh 3x}\,dx$$

$$= \tfrac{1}{3}\log_e(\sinh 3x).$$

(ii) $$\int \frac{\sin 2\theta + 3\cos 3\theta}{\cos 2\theta - 2\sin 3\theta}\,d\theta = \int \frac{-\frac{1}{2}\frac{d}{d\theta}(\cos 2\theta - 2\sin 3\theta)}{\cos 2\theta - 2\sin 3\theta}\,d\theta$$

$$= -\tfrac{1}{2}\log_e(\cos 2\theta - 2\sin 3\theta).$$

(iii) $$\int \frac{x^2 + 2x}{\sqrt{(x^3 + 3x^2 + 2)}}\,dx = \int \frac{\frac{1}{3}\frac{d}{dx}(x^3 + 3x^2 + 2)}{\sqrt{(x^3 + 3x^2 + 2)}}\,dx$$
$$= \tfrac{2}{3}\sqrt{(x^3 + 3x^2 + 2)}.$$

(iv) $$\int \frac{1 - 2t}{\sqrt{(3 + t - t^2)}}\,dt = \int \frac{\frac{d}{dt}(3 + t - t^2)}{\sqrt{(3 + t - t^2)}}\,dt = 2\sqrt{(3 + t - t^2)}.$$

6. Integration using partial fractions.

If the integrand is of the form $f(x)/\phi(x)$, and $f(x)$ and $\phi(x)$ are rational, integral, algebraical functions of x, and the integral does not come under the standard integrals already discussed, it will be necessary, where possible, to express the integrand in partial fractions before proceeding to determine its value.

Example 2.—Find the values of

$$\text{(i) } I = \int_0^2 \frac{3 + 2x}{1 + 2x}\,dx, \qquad \text{(ii) } I = \int_5^6 \frac{2 - x^2}{(x + 3)(x - 4)^2}\,dx,$$

to three decimal places. (I is usually used for an integral.)

(i) $$I = \int_0^2 \frac{(1 + 2x) + 2}{1 + 2x}\,dx = \int_0^2 \left(1 + \frac{2}{1 + 2x}\right)dx$$

(partial fractions by inspection)

$$= \Big[x + \log_e(1 + 2x)\Big]_0^2 = [(2 + \log_e 5) - (0 + \log_e 1)]$$

$$= 2 + \log_e 5 = 3{\cdot}609 \text{ to three decimal places.}$$

(ii) Let $$\frac{2 - x^2}{(x + 3)(x - 4)^2} = \frac{A}{x + 3} + \frac{B}{x - 4} + \frac{C}{(x - 4)^2}.$$

$$\therefore\ 2 - x^2 = A(x - 4)^2 + B(x + 3)(x - 4) + C(x + 3).$$

Since this is an identity,

putting $x = -3$, $\quad -7 = 49A, \ \therefore\ A = -\frac{1}{7};$

$x = 4$, $\quad -14 = 7C, \ \therefore\ C = -2;$

$x = 0$, $\quad 2 = 16A - 12B + 3C$

$\quad = -\frac{16}{7} - 12B - 6,$

$\therefore\ 12B = -\frac{72}{7}, \quad \therefore\ B = -\frac{6}{7}.$

$$\text{Hence } I = \int_5^6 \left\{ -\frac{1}{7(x+3)} - \frac{6}{7(x-4)} - \frac{2}{(x-4)^2} \right\} dx$$

$$= \left[-\tfrac{1}{7}\log_e(x+3) - \tfrac{6}{7}\log_e(x-4) + \frac{2}{x-4} \right]_5^6$$

$$= [(-\tfrac{1}{7}\log_e 9 - \tfrac{6}{7}\log_e 2 + 1) - (-\tfrac{1}{7}\log_e 8 - \tfrac{6}{7}\log_e 1 + 2)]$$

$$= -\tfrac{1}{7}\log_e 9 - \tfrac{6}{7}\log_e 2 + \tfrac{3}{7}\log_e 2 + 1 - 2$$

$$= -1 - \tfrac{1}{7}(\log_e 9 + 3\log_e 2) = -1 - 0{\cdot}611$$

$$= -1{\cdot}611 \text{ to three decimal places.}$$

7. Integration by substitution.

Certain integrals can only be determined by introducing a new variable which is a function of the original variable. Thus, if it be required to determine $\int f(x)\,dx$, and this be not possible by other means, it may be possible to obtain its value by substituting for x some suitable function of a new variable (e.g. $x = \sin z$).

Consider next what occurs when such a substitution is effected.

If $I = \int f(x)dx$, then $\dfrac{dI}{dx} = f(x)$.

Suppose $x = \phi(z)$ be the substitution used and $f[\phi(z)] = F(z)$.

Then $$\frac{dI}{dx} = F(z) \text{ and } \frac{dx}{dz} = \phi'(z).$$

Using the function of a function theorem,

$$\frac{dI}{dz} = \frac{dI}{dx}\cdot\frac{dx}{dz} = F(z)\phi'(z),$$

$$\therefore\ I = \int F(z)\phi'(z)\,dz.$$

Hence in effecting the substitution $x = \phi(z)$ in order to evaluate the integral I, x is replaced by $\phi(z)$ and dx by $\phi'(z)\,dz$ in the integral.

Integration by substitution will be dealt with more fully later on, and a table of useful substitutions given for special cases (p. 144).

N.B.—When dealing with definite integrals in which a substitution is necessary, it usually facilitates the working if the limits are changed to the corresponding values for the new variable as the substitution is effected.

Example 3.—Evaluate (i) $I = \int_0^2 \sqrt{(4 - x^2)}\, dx$, (ii) $I = \int_0^{\pi/2} \sin^2\theta \cos\theta\, d\theta$.

(i) Let $x = 2\sin\theta$, $\therefore dx = 2\cos\theta\, d\theta$. [This means that dx is replaced by $2\cos\theta\, d\theta$.]
When $x = 0$, $\theta = 0$, and when $x = 2$, $\theta = \pi/2$
(really use the substitution $\theta = \sin^{-1} x/2$).

$$\text{Substituting } I = \int_0^{\pi/2} \sqrt{(4 - 4\sin^2\theta)}\,.\,2\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} 2\cos\theta\,.\,2\cos\theta\, d\theta = \int_0^{\pi/2} 4\cos^2\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$= 2\Big[0 + \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi/2} = 2[\pi/2]$$

$$= \pi.$$

(ii) Let $z = \sin\theta$, $\therefore dz = \cos\theta\, d\theta$ [i.e. $\cos\theta\, d\theta$ is replaced by dz].
When $\theta = 0$, $z = 0$, and when $\theta = \pi/2$, $z = 1$.

$$\therefore I = \int_0^1 z^2\, dz = \Big[\tfrac{1}{3}z^3\Big]_0^1 = \tfrac{1}{3}.$$

8. Integration of standard types of trigonometric functions.

N.B.—Indefinite integrals taken throughout; starred results should be memorized.

(A) $I = \int \operatorname{cosec} x\, dx$.

$$I = \int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{1}{2}x\cos\frac{1}{2}x} = \int \frac{\frac{1}{2}\sec^2\frac{1}{2}x}{\tan\frac{1}{2}x}\, dx$$

$$= \int \frac{\frac{d}{dx}(\tan\frac{1}{2}x)}{\tan\frac{1}{2}x}\, dx = \log_e(\tan\tfrac{1}{2}x).^*$$

(B) $I = \int \sec x\, dx$.

$$I = \int \frac{dx}{\cos x} = \int \frac{dx}{\sin(x + \frac{1}{2}\pi)} = \log_e\{\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}x)\},^*$$

using the previous result with $(x + \frac{1}{2}\pi)$ replacing x, since $\frac{d}{dx}(x + \frac{1}{2}\pi) = 1$.

Aliter
$$I = \int \frac{dx}{\cos x} = \int \frac{dx}{\cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x}$$

$$= \int \frac{\sec^2\frac{1}{2}x\, dx}{1 - \tan^2\frac{1}{2}x}.$$

Let $\qquad z = \tan \frac{1}{2}x, \quad \therefore\ dz = \frac{1}{2}\sec^2 \frac{1}{2}x\, dx,$

and
$$I = \int \frac{2dz}{1 - z^2} = \int \left(\frac{1}{1 + z} + \frac{1}{1 - z}\right) dz$$
$$= \log_e(1 + z) - \log_e(1 - z) = \log_e \frac{1 + z}{1 - z}$$
$$= \log_e \frac{1 + \tan \frac{1}{2}x}{1 - \tan \frac{1}{2}x}.$$

Note.—These results are equivalent since
$$\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}x) = \frac{\tan \frac{1}{4}\pi + \tan \frac{1}{2}x}{1 - \tan \frac{1}{4}\pi \tan \frac{1}{2}x}$$
$$= \frac{1 + \tan \frac{1}{2}x}{1 - \tan \frac{1}{2}x}.$$

(C) $I = \displaystyle\int \frac{dx}{a \cos x + b \sin x}.$

Let
$$\left.\begin{aligned} a &= r \sin \alpha \\ b &= r \cos \alpha \end{aligned}\right\}, \qquad \begin{aligned} &\text{(i)} \\ &\text{(ii)} \end{aligned}$$

where r is positive and $0 \leqslant \alpha \leqslant 360°$.

(i)^2 + (ii)^2, $\qquad a^2 + b^2 = r^2, \quad \therefore\ r = \sqrt{(a^2 + b^2)};$

and from (i) $\div$ (ii), $\qquad \tan \alpha = a/b,$

where the quadrant in which α lies is indicated by (i) and (ii) with r positive.

Then $\displaystyle I = \int \frac{dx}{r \sin \alpha \cos x + r \cos \alpha \sin x} = \frac{1}{r}\int \frac{dx}{\sin(x + \alpha)}$
$$= \frac{1}{\sqrt{(a^2 + b^2)}} \int \operatorname{cosec}(x + \alpha)\, dx$$
$$= \frac{1}{\sqrt{(a^2 + b^2)}} \log_e\left(\tan \frac{x + \alpha}{2}\right),$$
where α is given above in (i) and (ii).

(D) $I_1 = \displaystyle\int \frac{dx}{a \sin x + b}$, and $I_2 = \displaystyle\int \frac{dx}{a \cos x + b}$.
$$I_1 = \int \frac{dx}{2a \sin \frac{1}{2}x \cos \frac{1}{2}x + b(\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x)}$$
$$= \int \frac{\sec^2 \frac{1}{2}x\, dx}{b + 2a \tan \frac{1}{2}x + b \tan^2 \frac{1}{2}x}.$$

Let $\tan \frac{1}{2}x = z, \quad \therefore \frac{1}{2} \sec^2 \frac{1}{2}x \, dx = dz,$

and $I_1 = \int \frac{2dz}{b + 2az + bz^2}$ (dealt with under algebraic functions, p. 139)

$$= \frac{2}{b} \int \frac{dz}{1 + (2a/b)z + z^2} \text{ (make coefficient of } z^2 \text{ unity)}$$

$$= \frac{2}{b} \int \frac{dz}{(1 - a^2/b^2) + (z + a/b)^2} \text{ (complete square of } z \text{ portion).}$$

There are two cases of this to be considered.

Case (a).—When $(1 - a^2/b^2)$ is *positive*,

$$I_1 = \frac{2}{b\sqrt{(1 - a^2/b^2)}} \tan^{-1} \frac{z + a/b}{\sqrt{(1 - a^2/b^2)}} = \frac{2}{\sqrt{(b^2 - a^2)}} \tan^{-1} \frac{bz + a}{\sqrt{(b^2 - a^2)}}$$

$$= \frac{2}{\sqrt{(b^2 - a^2)}} \tan^{-1} \frac{b \tan \frac{1}{2}x + a}{\sqrt{(b^2 - a^2)}}.$$

Case (b).—When $(1 - a^2/b^2)$ is *negative*,

$$I_1 = \frac{1}{b\sqrt{(a^2/b^2 - 1)}} \log_e \frac{z - \sqrt{(a^2/b^2 - 1)}}{z + \sqrt{(a^2/b^2 - 1)}}$$

$$\left(\text{using value of } \int \frac{dy}{y^2 - \alpha^2}\right),$$

i.e. $$I_1 = \frac{1}{\sqrt{(a^2 - b^2)}} \log_e \frac{b \tan \frac{1}{2}x - \sqrt{(a^2 - b^2)}}{b \tan \frac{1}{2}x + \sqrt{(a^2 - b^2)}}.$$

$$I_2 = \int \frac{dx}{a(\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x) + b(\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x)}$$

$$= \int \frac{\sec^2 \frac{1}{2}x \, dx}{(a + b) + (b - a) \tan^2 \frac{1}{2}x}.$$

Let $\tan \frac{1}{2}x = z, \quad \therefore \frac{1}{2} \sec^2 \frac{1}{2}x \, dx = dz,$

and $$I_2 = \int \frac{2dz}{(a + b) + (b - a)z^2} = \frac{2}{b - a} \int \frac{dz}{\dfrac{a + b}{b - a} + z^2}.$$

There are two cases of this to be considered.

Case (*a*).—When $(a + b)/(b - a)$ is positive,

$$I_2 = \frac{2}{b-a} \cdot \frac{1}{\sqrt{\left(\frac{a+b}{b-a}\right)}} \tan^{-1}\left\{z\Big/\sqrt{\left(\frac{a+b}{b-a}\right)}\right\}$$

$$= \frac{2}{\sqrt{(b^2-a^2)}} \tan^{-1}\left\{\sqrt{\left(\frac{b-a}{b+a}\right)} \cdot \tan\frac{x}{2}\right\}.$$

Case (*b*).—When $(a + b)/(b - a)$ is negative,

$$I_2 = \frac{1}{b-a} \cdot \frac{1}{\sqrt{\left(\frac{a+b}{a-b}\right)}} \log_e \frac{z - \sqrt{\left(\frac{a+b}{a-b}\right)}}{z + \sqrt{\left(\frac{a+b}{a-b}\right)}}$$

$$\left(\text{using} \int \frac{dy}{y^2 - \alpha^2}\right),$$

i.e. $$I_2 = \frac{-1}{\sqrt{(a^2-b^2)}} \log_e \frac{\sqrt{(a-b)}\tan\frac{1}{2}x - \sqrt{(a+b)}}{\sqrt{(a-b)}\tan\frac{1}{2}x + \sqrt{(a+b)}}.$$

Example 4.—Find the indefinite integrals

$$(a)\ I = \int \frac{dx}{2 + 3\cos x}, \quad (b)\ I = \int \frac{\sin x}{2 + \sin x}\,dx.$$

(*a*) $$I = \int \frac{dx}{2(\cos^2\frac{1}{2}x + \sin^2\frac{1}{2}x) + 3(\cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x)} = \int \frac{dx}{5\cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x}$$

$$= \int \frac{\sec^2\frac{1}{2}x\,dx}{5 - \tan^2\frac{1}{2}x}.$$

Let $\tan\frac{1}{2}x = z, \quad \therefore \frac{1}{2}\sec^2\frac{1}{2}x\,dx = dz,$

and $$I = \int \frac{2\,dz}{5 - z^2} = \frac{1}{\sqrt{5}} \log_e \frac{\sqrt{5} + z}{\sqrt{5} - z}.$$

$$= \frac{1}{\sqrt{5}} \log_e \frac{\sqrt{5} + \tan\frac{1}{2}x}{\sqrt{5} - \tan\frac{1}{2}x}.$$

(*b*) $$I = \int \frac{(2 + \sin x) - 2}{2 + \sin x}\,dx \text{ (similar to partial fractions)}$$

$$= \int \left(1 - \frac{2}{2 + \sin x}\right)dx$$

$$= x - 2\int \frac{dx}{2(\cos^2\frac{1}{2}x + \sin^2\frac{1}{2}x) + 2\sin\frac{1}{2}x\cos\frac{1}{2}x}$$

$$= x - 2\int \frac{\frac{1}{2}\sec^2\frac{1}{2}x\,dx}{1 + \tan\frac{1}{2}x + \tan^2\frac{1}{2}x}$$

$$= x - 2I_1, \text{ where } I_1 = \int \frac{\frac{1}{2}\sec^2\frac{1}{2}x\,dx}{1 + \tan\frac{1}{2}x + \tan^2\frac{1}{2}x}.$$

Let $$\tan \tfrac{1}{2}x = z \text{ in } I_1, \quad \therefore \tfrac{1}{2}\sec^2 \tfrac{1}{2}x\, dx = dz,$$

and
$$I_1 = \int \frac{dz}{1+z+z^2} = \int \frac{dz}{(z+\frac{1}{2})^2 + \frac{3}{4}}$$
$$= \frac{1}{\sqrt{(\frac{3}{4})}} \tan^{-1} \frac{z+\frac{1}{2}}{\sqrt{(\frac{3}{4})}} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{2z+1}{\sqrt{3}}$$
$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2\tan\frac{1}{2}x + 1}{\sqrt{3}},$$
$$\therefore I = x - \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{2\tan\frac{1}{2}x+1}{\sqrt{3}}\right).$$

(E) $I = \int dx \Big/ (a^2 \sin^2 x \pm b^2 \cos^2 x)$.

$$I = \frac{1}{a^2}\int \frac{\sec^2 x\, dx}{\tan^2 x \pm b^2/a^2}.$$

Let $$\tan x = z, \quad \therefore \sec^2 x\, dx = dz,$$

and
$$I = \frac{1}{a^2}\int \frac{dz}{z^2 \pm b^2/a^2}.$$

With the *positive sign*,
$$I = \frac{1}{a^2}\frac{a}{b} \tan^{-1} \frac{z}{b/a}$$
$$= \frac{1}{ab} \tan^{-1}\left(\frac{a \tan x}{b}\right).$$

With the *negative sign*,
$$I = \frac{1}{a^2}\frac{a}{2b} \log_e \frac{z - b/a}{z + b/a}$$
$$= \frac{1}{2ab} \log_e \frac{a\tan x - b}{a \tan x + b}$$
(using $\tan^2 x > b^2/a^2$).

(F) $I_1 = \int \sin^2 x\, dx, \quad I_2 = \int \cos^2 x\, dx.$

$$I_1 = \tfrac{1}{2}\int (1 - \cos 2x) dx = \tfrac{1}{2}(x - \tfrac{1}{2}\sin 2x).$$
$$I_2 = \tfrac{1}{2}\int (1 + \cos 2x) dx = \tfrac{1}{2}(x + \tfrac{1}{2}\sin 2x).$$

(G) $I = \int \sin^m x \cos^n x\, dx$, where m and n are positive integers. There are three cases, as follows, of this integral to be considered.

Case (a).—m odd (n odd, even, or zero).

$$I = \int \sin^{m-1} x \cos^n x \sin x\, dx = \int (1 - \cos^2 x)^{(m-1)/2} \cos^n x \sin x\, dx.$$

Let $\cos x = z, \quad \therefore \ -\sin x\, dx = dz,$

and $$I = -\int (1 - z^2)^{(m-1)/2} z^n\, dz.$$

Since $(m-1)$ is even, $(m-1)/2$ is a positive integer, and hence $(1-z^2)^{(m-1)/2}$ can be expanded as a finite series by means of the binomial theorem. Thus, the last integral can be found, and, by replacing z in the result by $\cos x$, the value of the original integral can be determined.

Example 5.—Find $I = \int \sin^3 x \cos^2 x\, dx$.

$$I = \int \sin^2 x \cos^2 x \,.\, \sin x\, dx = \int (1 - \cos^2 x) \cos^2 x \,.\, \sin x\, dx.$$

Let $z = \cos x, \quad \therefore \ dz = -\sin x\, dx,$

and $$I = -\int (1 - z^2) z^2\, dz = \int (z^4 - z^2) dz$$
$$= \tfrac{1}{5} z^5 - \tfrac{1}{3} z^3 + C = \tfrac{1}{5} \cos^5 x - \tfrac{1}{3} \cos^3 x + C.$$

Case (*b*).—n odd (m odd, even, or zero).

$$I = \int \sin^m x \cos^{n-1} x \cos x\, dx = \int \sin^m x (1 - \sin^2 x)^{(n-1)/2} \cos x\, dx.$$

Let $z = \sin x, \quad \therefore \ \cos x\, dx = dz,$

and $$I = \int z^m (1 - z^2)^{(n-1)/2}\, dz.$$

Since $(n-1)/2$ is a positive integer $(1 - z^2)^{(n-1)/2}$ can be expanded as a finite series by the binomial theorem and the integral finally determined in terms of x.

Example 6.—Evaluate $I = \int_0^{\pi/2} \cos^5 x\, dx$.

Let $\sin x = z, \quad \therefore \ \cos x\, dx = dz.$

When $x = 0$, $z = 0$, and when $x = \pi/2$, $z = 1$.

$$\therefore \ I = \int_0^1 \cos^4 x\, dz = \int_0^1 (1 - \sin^2 x)^2\, dz$$
$$= \int_0^1 (1 - z^2)^2\, dz = \int_0^1 (1 - 2z^2 + z^4)\, dz$$
$$= \Big[z - \tfrac{2}{3} z^3 + \tfrac{1}{5} z^5\Big]_0^1 = 1 - \tfrac{2}{3} + \tfrac{1}{5} = \tfrac{8}{15}.$$

Case (*c*).—Both m and n even.

In this case $\sin^m x \cos^n x$ must first be expressed in terms of the sines and cosines of the multiple angles as a linear expression. This can be done by using some or all of the following trigonometric identities:

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$; $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$;
$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$; $\sin\theta_1\cos\theta_2 = \frac{1}{2}\{\sin(\theta_1 + \theta_2) + \sin(\theta_1 - \theta_2)\}$;
$\cos\theta_1\cos\theta_2 = \frac{1}{2}\{\cos(\theta_1 + \theta_2) + \cos(\theta_1 - \theta_2)\}$;
$\sin\theta_1\sin\theta_2 = \frac{1}{2}\{\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2)\}$;

or the exponential values of $\sin\theta$ and $\cos\theta$, namely,

$$\cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta});\quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}),$$

and then the integration can be dealt with.

Example 7.—Find $I = \int \sin^4 x \cos^2 x\, dx$.

$$\begin{aligned}
I &= \int (\sin x\cos x)^2 . \sin^2 x\, dx \\
&= \int (\tfrac{1}{2}\sin 2x)^2 . \tfrac{1}{2}(1 - \cos 2x)\, dx \\
&= \tfrac{1}{8}\int \tfrac{1}{2}(1 - \cos 4x)(1 - \cos 2x)\, dx \\
&= \tfrac{1}{16}\int (1 - \cos 2x - \cos 4x + \cos 2x\cos 4x)\, dx \\
&= \tfrac{1}{16}\int (1 - \cos 2x - \cos 4x + \tfrac{1}{2}[\cos 6x + \cos 2x])\, dx \\
&= \tfrac{1}{32}\int \{2 - \cos 2x - 2\cos 4x + \cos 6x\}\, dx \\
&= \tfrac{1}{32}[2x - \tfrac{1}{2}\sin 2x - \tfrac{1}{2}\sin 4x + \tfrac{1}{6}\sin 6x] + C.
\end{aligned}$$

(H) $I_1 = \int \sin px\cos qx\, dx$, $I_2 = \int \cos px\cos qx\, dx$,

$$I_3 = \int \sin px\sin qx\, dx.$$

Using the trigonometric identities in (G),

$$\begin{aligned}
I_1 &= \tfrac{1}{2}\int \{\sin(p+q)x + \sin(p-q)x\}\, dx \\
&= -\tfrac{1}{2}\left\{\frac{\cos(p+q)x}{p+q} + \frac{\cos(p-q)x}{p-q}\right\}. \\
I_2 &= \tfrac{1}{2}\int \{\cos(p+q)x + \cos(p-q)x\}\, dx \\
&= \tfrac{1}{2}\left\{\frac{\sin(p+q)x}{p+q} + \frac{\sin(p-q)x}{p-q}\right\}. \\
I_3 &= \tfrac{1}{2}\int \{\cos(p-q)x - \cos(p+q)x\}\, dx \\
&= \tfrac{1}{2}\left\{\frac{\sin(p-q)x}{p-q} - \frac{\sin(p+q)x}{p+q}\right\}.
\end{aligned}$$

(I) (a) $I_1 = \int \tan^p x \sec^2 x \, dx$,

(b) $I_2 = \int \tan^m x \, dx$, where m is a positive integer greater than 1.

(a) Let $\tan x = z, \quad \therefore \sec^2 x \, dx = dz,$

and
$$I_1 = \int z^p \, dz = \frac{z^{p+1}}{p+1} = \frac{\tan^{p+1} x}{p+1}.$$

(b)
$$\begin{aligned} I_2 &= \int \tan^{m-2} x \tan^2 x \, dx \\ &= \int \tan^{m-2} x(\sec^2 x - 1) \, dx \\ &= \int \tan^{m-2} x \sec^2 x \, dx - \int \tan^{m-2} x \, dx \\ &= \frac{1}{m-1} \tan^{m-1} x - \int \tan^{m-4}(\sec^2 x - 1) \, dx \quad (\text{if } m > 3). \end{aligned}$$

This process of replacing $\tan^2 x$ by $(\sec^2 x - 1)$ can be continued until the final integral is either $\int \tan x \, dx = \log_e (\sec x)$, or $\int dx = x$. Thus the original integral can be found.

N.B.—A similar process can be used for $\int \cot^n x \, dx$, where n is a positive integer greater than unity, by using $\cot^2 x = \operatorname{cosec}^2 x - 1$.

Example 8.—Evaluate (i) $I = \int_0^{\pi/4} \tan^6 \theta \, d\theta$, (ii) $I = \int \operatorname{cosec}^4 \theta \, d\theta$.

(i)
$$\begin{aligned} I &= \int_0^{\pi/4} \tan^4 \theta(\sec^2 \theta - 1) \, d\theta \\ &= \left[\tfrac{1}{5} \tan^5 \theta\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^4 \theta \, d\theta = \tfrac{1}{5} - \int_0^{\pi/4} (\sec^2 \theta - 1) \tan^2 \theta \, d\theta \\ &= \tfrac{1}{5} - \tfrac{1}{3}\left[\tan^3 \theta\right]_0^{\pi/4} + \int_0^{\pi/4} \tan^2 \theta \, d\theta \\ &= \tfrac{1}{5} - \tfrac{1}{3} + \int_0^{\pi/4} (\sec^2 \theta - 1) \, d\theta \\ &= \tfrac{1}{5} - \tfrac{1}{3} + \left[\tan \theta - \theta\right]_0^{\pi/4} \\ &= \tfrac{1}{5} - \tfrac{1}{3} + 1 - \pi/4 = \tfrac{13}{15} - \pi/4. \end{aligned}$$

(ii)
$$\begin{aligned} I &= \int (1 + \cot^2 \theta) \operatorname{cosec}^2 \theta \, d\theta = \int \operatorname{cosec}^2 \theta \, d\theta + \int \cot^2 \theta \operatorname{cosec}^2 \theta \, d\theta \\ &= -\cot \theta - \tfrac{1}{3} \cot^3 \theta + C. \end{aligned}$$

N.B.—Similar methods to those for the trigonometric functions

can be adopted for the determination of integrals involving the corresponding hyperbolic functions.

Thus $\int \tanh x\, dx = \int \frac{\sinh x}{\cosh x}\, dx = \int \frac{d/dx\,(\cosh x)}{\cosh x}\, dx = \log_e(\cosh x)$.

Also if $I = \int \operatorname{sech} u\, du$, then

$$I = \int \frac{du}{\cosh u} = \int \frac{du}{\cosh^2 \frac{1}{2}u + \sinh^2 \frac{1}{2}u} = \int \frac{\operatorname{sech}^2 \frac{1}{2}u}{1 + \tanh^2 \frac{1}{2}u}\, du.$$

Let $\qquad z = \tanh \frac{1}{2}u, \quad \therefore\ dz = \frac{1}{2}\operatorname{sech}^2 \frac{1}{2}u\, du,$

and $\qquad I = \int \frac{2}{1+z^2}\, dz = 2\tan^{-1} z = 2\tan^{-1}(\tanh \frac{1}{2}u).$

9. Integration of standard types of algebraic functions.

In what follows, P, Q, p, q are constants, and the quadratic functions shown cannot be factorized.

(i) $I = \int \frac{Px + Q}{x^2 + px + q}\, dx.$

Making the x portion of the numerator into a multiple of the differential coefficient of the denominator,

$$I = \int \frac{\frac{1}{2}P(2x+p) + (Q - \frac{1}{2}pP)}{x^2 + px + q}\, dx$$

$$= \frac{1}{2}P \int \frac{2x+p}{x^2+px+q}\, dx + (Q - \frac{1}{2}pP)\int \frac{dx}{x^2+px+q}$$

$$= \frac{1}{2}P \log_e (x^2+px+q) + (Q - \frac{1}{2}pP)\int \frac{dx}{(x+p/2)^2 + q - p^2/4}$$

$$= \frac{1}{2}P \log_e (x^2+px+q) + \frac{Q - \frac{1}{2}pP}{\sqrt{(q-p^2/4)}} \tan^{-1} \frac{x+p/2}{\sqrt{(q-p^2/4)}} + C$$

(when $q - p^2/4 > 0$),

and $$= \frac{1}{2}P \log_e (x^2+px+q)$$

$$+ \frac{Q - \frac{1}{2}pP}{2\sqrt{(p^2/4 - q)}} \log_e \frac{x + \frac{1}{2}p - \sqrt{(p^2/4-q)}}{x + \frac{1}{2}p + \sqrt{(p^2/4-q)}} + C$$

(when $q - p^2/4 < 0$).

Example 9.—Find $I = \int \dfrac{11 - 9x}{3x^2 + 4x + 15}\, dx.$

$$I = \int \frac{(11 + 6) - \frac{9}{6}(6x + 4)}{3x^2 + 4x + 15}\, dx$$

$$= \frac{17}{3}\int \frac{dx}{x^2 + \frac{4}{3}x + 5} - \frac{3}{2}\int \frac{6x + 4}{3x^2 + 4x + 15}\, dx$$

$$= \frac{17}{3}\int \frac{dx}{(x + \frac{2}{3})^2 + \frac{41}{9}} - \frac{3}{2}\log_e(3x^2 + 4x + 15)$$

$$= \frac{17}{3} \times \frac{1}{\sqrt{\frac{41}{9}}}\tan^{-1}\frac{x + \frac{2}{3}}{\sqrt{\frac{41}{9}}} - \frac{3}{2}\log_e(3x^2 + 4x + 15) + C$$

$$= \frac{17}{\sqrt{41}}\tan^{-1}\frac{3x + 2}{\sqrt{41}} - \frac{3}{2}\log_e(3x^2 + 4x + 15) + C.$$

(ii) $I = \displaystyle\int \frac{Px + Q}{\sqrt{(x^2 + px + q)}}\, dx.$

$$I = \int \frac{(2x + p)P/2 + Q - Pp/2}{\sqrt{(x^2 + px + q)}}\, dx$$

$$= P\int \frac{\frac{1}{2}(2x + p)}{\sqrt{(x^2 + px + q)}}\, dx + \left(Q - \frac{pP}{2}\right)\int \frac{dx}{\sqrt{\{(x + p/2)^2 + (q - p^2/4)\}}}$$

$$= P\sqrt{(x^2 + px + q)} + \left(Q - \frac{pP}{2}\right)\sinh^{-1}\frac{x + p/2}{\sqrt{(q - p^2/4)}} + C$$

(with $q > p^2/4$),

and

$$= P\sqrt{(x^2 + px + q)} + \left(Q - \frac{pP}{2}\right)\cosh^{-1}\frac{x + p/2}{\sqrt{(p^2/4 - q)}} + C$$

(with $q < p^2/4$).

Example 10.—Find $I = \int \dfrac{1 - 8v}{\sqrt{(16v^2 + 40v - 47)}}\, dv.$

$$I = \int \frac{-\frac{1}{4}(32v + 40) + 11}{\sqrt{(16v^2 + 40v - 47)}}\, dv$$

$$= -\frac{1}{2}\int \frac{\frac{1}{2}(32v + 40)}{\sqrt{(16v^2 + 40v - 47)}}\, dv + \frac{11}{4}\int \frac{dv}{\sqrt{(v^2 + \frac{5}{2}v - \frac{47}{16})}}$$

(*N.B.*—Coefficient of v^2 made unity.)

$$= -\frac{1}{2}\sqrt{(16v^2 + 40v - 47)} + \frac{11}{4}\int \frac{dv}{\sqrt{\{(v + \frac{5}{4})^2 - \frac{9}{2}\}}}$$

$$= -\frac{1}{2}\sqrt{(16v^2 + 40v - 47)} + \frac{11}{4}\cosh^{-1}\frac{v + \frac{5}{4}}{\sqrt{\frac{9}{2}}} + C$$

$$= \frac{11}{4}\cosh^{-1}\frac{(4v + 5)\sqrt{2}}{12} - \frac{1}{2}\sqrt{(16v^2 + 40v - 47)} + C.$$

(iii) $I = \int \frac{Px + Q}{\sqrt{(q + px - x^2)}} dx$

$$= \int \frac{-(-2x + p)P/2 + (Q + pP/2)}{\sqrt{(q + px - x^2)}} dx$$

$$= -P \int \frac{\frac{1}{2}(-2x + p)}{\sqrt{(q + px - x^2)}} dx + \left(Q + \frac{pP}{2}\right) \int \frac{dx}{\sqrt{\{(q + p^2/4) - (x - p/2)^2\}}}$$

$$= -P\sqrt{(q + px - x^2)} + \left(Q + \frac{pP}{2}\right) \sin^{-1} \frac{x - p/2}{\sqrt{(q + p^2/4)}} + C.$$

Example 11.—Find $I = \int \frac{12x - 11}{\sqrt{(4 + 7x - x^2)}} dx.$

$$I = \int \frac{-6(-2x + 7) + 31}{\sqrt{(4 + 7x - x^2)}} dx$$

$$= -12 \int \frac{\frac{1}{2}(-2x + 7)}{\sqrt{(4 + 7x - x^2)}} dx + 31 \int \frac{dx}{\sqrt{\{\frac{65}{4} - (x - \frac{7}{2})^2\}}}$$

$$= -12\sqrt{(4 + 7x - x^2)} + 31 \sin^{-1} \frac{x - \frac{7}{2}}{\sqrt{65}/2} + C$$

$$= -12\sqrt{(4 + 7x - x^2)} + 31 \sin^{-1} \frac{2x - 7}{\sqrt{65}} + C.$$

(iv) $I = \int \sqrt{(x^2 + px + q)}\, dx.$

$$I = \int \sqrt{\{(x + p/2)^2 + (q - p^2/4)\}}\, dx = \int \sqrt{\{(x + p/2)^2 \pm c^2\}}\, dx$$

(where $c^2 = |\, q - p^2/4 \,|$).

Case (*a*).—Positive sign, i.e. $q > p^2/4$.

Let $x + p/2 = c \sinh\theta, \quad \therefore\ dx = c \cosh\theta\, d\theta,$

and
$$I = \int \sqrt{c^2 \sinh^2\theta + c^2} \,.\, c \cosh\theta\, d\theta$$

$$= c^2 \int \cosh^2\theta\, d\theta \qquad (1 + \sinh^2\theta = \cosh^2\theta)$$

$$= \frac{c^2}{2} \int (1 + \cosh 2\theta)\, d\theta$$

$$= \frac{c^2}{2} [\theta + \tfrac{1}{2} \sinh 2\theta] + C$$

$$= \frac{c^2}{2} [\theta + \sinh\theta \cosh\theta] + C$$

$$= \frac{c^2}{2}[\theta + \sinh\theta\sqrt{(1+\sinh^2\theta)}] + C$$

$$= \frac{c^2}{2}\left[\sinh^{-1}\frac{x+p/2}{c} + \frac{x+p/2}{c}\sqrt{1+\frac{(x+p/2)^2}{c^2}}\right] + C$$

$$= \frac{q-p^2/4}{2}\sinh^{-1}\frac{x+p/2}{\sqrt{(q-p^2/4)}}$$

$$+ \tfrac{1}{2}(x+p/2)\sqrt{q-p^2/4+(x+p/2)^2} + C$$

$$= \frac{q-p^2/4}{2}\sinh^{-1}\frac{x+p/2}{\sqrt{(q-p^2/4)}} + \tfrac{1}{2}(x+p/2)\sqrt{x^2+px+q} + C.$$

Case (*b*).—Negative sign, i.e. $q < p^2/4$.

Let $x + p/2 = c\cosh\theta, \quad \therefore\ dx = c\sinh\theta\, d\theta,$

and $I = \int\sqrt{c^2\cosh^2\theta - c^2}\,.\,c\sinh\theta\,d\theta = c^2\int\sinh^2\theta\,d\theta$

$$= \frac{c^2}{2}\int(\cosh 2\theta - 1)\,d\theta$$

$$= \frac{c^2}{2}[\tfrac{1}{2}\sinh 2\theta - \theta] + C = \frac{c^2}{2}[\sinh\theta\cosh\theta - \theta] + C$$

$$= \tfrac{1}{2}[c^2\sinh\theta\cosh\theta - c^2\theta] + C$$

$$= \tfrac{1}{2}\left[(x+p/2)\sqrt{c^2\cosh^2\theta - c^2} - c^2\cosh^{-1}\frac{x+p/2}{c}\right] + C$$

$$= \tfrac{1}{2}\left[(x+p/2)\sqrt{x^2+px+q} - (p^2/4-q)\cosh^{-1}\frac{x+p/2}{\sqrt{(p^2/4-q)}}\right] + C.$$

Example 12.—Find $I = \int\sqrt{(2x^2 - 2x - 1)}\,dx.$

$$I = \sqrt{2}\int\sqrt{(x^2 - x - \tfrac{1}{2})}\,dx = \sqrt{2}\int\sqrt{\{(x-\tfrac{1}{2})^2 - \tfrac{3}{4}\}}\,dx.$$

Let $(x - \tfrac{1}{2}) = \frac{\sqrt{3}}{2}\cosh\theta, \quad \therefore\ dx = \frac{\sqrt{3}}{2}\sinh\theta,$

and $I = \sqrt{2}\int\sqrt{(\tfrac{3}{4}\cosh^2\theta - \tfrac{3}{4})}\,.\,\frac{\sqrt{3}}{2}\sinh\theta\,d\theta$

$$= \frac{3\sqrt{2}}{4}\int\sinh^2\theta\,d\theta = \frac{3\sqrt{2}}{8}\int(\cosh 2\theta - 1)\,d\theta$$

$$= \frac{3\sqrt{2}}{8}\left[\frac{\sinh 2\theta}{2} - \theta\right] + C$$

$$= \frac{3\sqrt{2}}{8}[\sinh\theta\cosh\theta - \theta] + C$$

$$= \frac{3\sqrt{2}}{8}[\cosh\theta\sqrt{(\cosh^2\theta - 1)} - \theta] + C$$

$$= \frac{3\sqrt{2}}{8}\left[\frac{(2x-1)}{\sqrt{3}}\sqrt{\left\{\frac{(2x-1)^2}{3} - 1\right\}} - \cosh^{-1}\frac{2x-1}{\sqrt{3}}\right] + C$$

$$= \frac{3\sqrt{2}}{8}\left[\frac{(2x-1)}{3}\sqrt{(4x^2 - 4x - 2)} - \cosh^{-1}\frac{2x-1}{\sqrt{3}}\right] + C$$

$$= \left[\frac{(2x-1)}{4}\sqrt{(2x^2 - 2x - 1)} - \frac{3\sqrt{2}}{8}\cosh^{-1}\frac{2x-1}{\sqrt{3}}\right] + C.$$

(v) Find $I = \int\sqrt{(q + px - x^2)}\,dx.$

$$I = \int\sqrt{\{(q + p^2/4) - (x - p/2)^2\}}\,dx.$$

Let $$x - p/2 = \sqrt{q + (p^2/4)}\sin\theta$$

($q + p^2/4$ must be positive if dealing with real quantities),

$$\therefore\ dx = \sqrt{q + (p^2/4)}\cos\theta\,d\theta,$$

and

$$I = \int\sqrt{(q + p^2/4)(1 - \sin^2\theta)}\sqrt{q + (p^2/4)}\cos\theta\,d\theta$$

$$= (q + p^2/4)\int\cos^2\theta\,d\theta = \tfrac{1}{2}(q + p^2/4)\int(1 + \cos 2\theta)\,d\theta$$

$$= \tfrac{1}{2}(q + p^2/4)[\theta + \tfrac{1}{2}\sin 2\theta] + C = \tfrac{1}{2}(q + p^2/4)[\theta + \sin\theta\cos\theta] + C$$

$$= \tfrac{1}{2}(q + p^2/4)[\theta + \sin\theta\sqrt{(1 - \sin^2\theta)}] + C$$

$$= \tfrac{1}{2}(q + p^2/4)\left[\sin^{-1}\frac{x - p/2}{\sqrt{(q + p^2/4)}} + \frac{x - p/2}{\sqrt{(q + p^2/4)}}\sqrt{1 - \frac{(x - p/2)^2}{q + p^2/4}}\right] + C$$

$$= \tfrac{1}{2}\left[(q + p^2/4)\sin^{-1}\frac{x - p/2}{\sqrt{(q + p^2/4)}} + (x - p/2)\sqrt{(q + px - x^2)}\right] + C.$$

Example **13.—Evaluate** $I = \int_0^a \sqrt{(2ax - x^2)}\,dx.$

$$I = \int_0^a \sqrt{\{a^2 - (x - a)^2\}}\,dx.$$

Let $$(x - a) = a\sin\theta, \quad \left[\text{really using } \theta = \sin^{-1}\frac{x - a}{a}\right].$$

$$\therefore\ dx = a\cos\theta\,d\theta.$$

When $x = a$, $\theta = 0$, and when $x = 0$, $\theta = -\pi/2$.

$$\therefore\ I = \int_{-\pi/2}^{0} \sqrt{a^2 - a^2 \sin^2 \theta}\,.\,a \cos\theta\, d\theta$$

$$= a^2 \int_{-\pi/2}^{0} \cos^2\theta\, d\theta = \frac{a^2}{2}\int_{-\pi/2}^{0} (1 + \cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_{-\pi/2}^{0} = \frac{a^2}{2}\Big[0 - \Big(\frac{-\pi}{2}\Big)\Big]$$

$$= \frac{\pi a^2}{4}.$$

10. Substitutions.

The following is a table of *suggested* substitutions for certain types of integrals, and the underlined substitutions are the more usual ones used when more than one substitution is given.

Type of integrand	Suggested substitution
Expression containing $\sqrt{a^2 - x^2}$	$x = \underline{a \sin\theta}$, $a\cos\theta$, $\underline{a \tanh\theta}$, $a\,\mathrm{sech}\,\theta$
Expression containing $\sqrt{a^2 + x^2}$	$x = \underline{a \sinh\theta}$, $a\,\mathrm{cosech}\,\theta$, $\underline{a\tan\theta}$, $a\cot\theta$
Expression containing $\sqrt{x^2 - a^2}$	$x = \underline{a\cosh\theta}$, $a\coth\theta$, $\underline{a\sec\theta}$, $a\,\mathrm{cosec}\,\theta$
Expression containing fractional powers of x	$x = z^r$, where r is the L.C.M. of the denominators of the fractional indices
$x\,f(x^2)$	$x^2 = z$
Function of x and $\sqrt{px + q}$	$\sqrt{px+q} = z$, i.e. $px + q = z^2$
Function of x and $\sqrt{(x-a)(x-b)}$	$\sqrt{x-b} = z\sqrt{x-a}$
Function of x and $\sqrt{(b-x)(x-a)}$	$\sqrt{b-x} = z\sqrt{x-a}$
Function of x and $\sqrt{x^2 + bx + c}$	$x + \sqrt{x^2+bx+c} = z$

N.B.—When dealing with $\int (x^2 + a^2)^n\, dx$, where n is positive, it is advisable to use $x = a\sinh\theta$, and when n is negative, use $x = a\tan\theta$. Any integral not given above and requiring a substitution must be experimented with.

Theorem.—To find $I = \displaystyle\int \frac{dx}{(x \pm a)\sqrt{(x^2 + px + q)}}$,

where $(x^2 + px + q)$ cannot be factorized.

Let $\quad x + a = 1/z, \quad \therefore\ dx = -dz/z^2,$ and $x = 1/z - a.$

$$\therefore\ I = \int \sqrt{\left\{\frac{z^2}{(1/z - a)^2 + p(1/z - a) + q}\right\}} \cdot \left(-\frac{1}{z^2}\right) dz$$

$$= -\int \frac{dz}{\sqrt{\{(az - 1)^2 - p(az - 1)z + qz^2\}}}$$

$$= -\int \frac{dz}{\sqrt{\{z^2(a^2 - ap + q) + z(p - 2a) + 1\}}}.$$

This last integral is of a type which has already been considered and hence can be determined. The integral could also have been found by using the last substitution shown in the table.

Theorem.—To find $I = \int \dfrac{dx}{x(px^m + q)}$.

Let $$x^m = \frac{1}{z}, \quad \therefore\ mx^{m-1}\, dx = -\frac{1}{z^2}\, dz,$$

i.e. $$\frac{m}{xz}\, dx = -\frac{1}{z^2}\, dz, \quad \therefore\ \frac{dx}{x} = -\frac{1}{mz}\, dz.$$

$$\therefore\ I = \int \frac{1}{(p/z + q)}\left(-\frac{1}{mz}\right) dz = -\frac{1}{m}\int \frac{dz}{p + qz}$$

$$= -\frac{1}{mq}\log_e (p + qz) + C$$

$$= -\frac{1}{mq}\log_e \frac{px^m + q}{x^m} + C.$$

Example 14.—Find $I = \int (x^2 + a^2)^{3/2}\, dx.$

Let $$x = a \sinh \theta, \quad \therefore\ dx = a \cosh \theta\, d\theta,$$

and $$I = \int a^4 \cosh^4 \theta\, d\theta = \frac{a^4}{4}\int (1 + \cosh 2\theta)^2\, d\theta$$

$$= \frac{a^4}{4}\int (1 + 2\cosh 2\theta + \cosh^2 2\theta)\, d\theta$$

$$= \frac{a^4}{4}\int \{1 + 2\cosh 2\theta + \tfrac{1}{2}[1 + \cosh 4\theta]\}\, d\theta$$

$$= \frac{a^4}{8}\int (3 + 4\cosh 2\theta + \cosh 4\theta)\, d\theta$$

$$= \frac{a^4}{8}[3\theta + 2\sinh 2\theta + \tfrac{1}{4}\sinh 4\theta] + C$$

$$= \frac{a^4}{8}[3\theta + 4\sinh\theta\cosh\theta + \sinh\theta\cosh\theta\cosh 2\theta] + C$$

$$= \frac{a^4}{8}[3\theta + 4\sinh\theta\cosh\theta + \sinh\theta\cosh\theta\,(1 + 2\sinh^2\theta)] + C$$

$$= \frac{a^4}{8}[3\theta + 4\sinh\theta\sqrt{(1 + \sinh^2\theta)} + \sinh\theta(1 + 2\sinh^2\theta)\sqrt{(1 + \sinh^2\theta)}] + C$$

$$= \frac{3a^4}{8}\sinh^{-1}\frac{x}{a} + \frac{a^2x}{2}\sqrt{(a^2 + x^2)} + \tfrac{1}{8}x(a^2 + 2x^2)\sqrt{(a^2 + x^2)} + C.$$

Example 15.—Find $I = \int \frac{dx}{(a^2 - x^2)^{3/2}}$.

Let $x = a\sin\theta, \quad \therefore\ dx = a\cos\theta\,d\theta,$

and
$$I = \int \frac{a\cos\theta\,d\theta}{(a^2 - a^2\sin^2\theta)^{3/2}} = \frac{1}{a^2}\int\frac{d\theta}{\cos^2\theta}$$

$$= \frac{1}{a^2}\int\sec^2\theta\,d\theta = \frac{1}{a^2}\tan\theta + C$$

$$= \frac{1}{a^2}\,\frac{\sin\theta}{\sqrt{(1 - \sin^2\theta)}} + C = \frac{x/a}{a^2\sqrt{(1 - x^2/a^2)}} + C$$

$$= \frac{x}{a^2\sqrt{(a^2 - x^2)}} + C.$$

Example 16.—Evaluate $I = \int_0^a \frac{dx}{(a^2 + x^2)^2}$.

Let $x = a\tan\theta, \quad \therefore\ d\theta = a\sec^2\theta\,d\theta \qquad (\theta = \tan^{-1}x/a).$

When $x = 0$, $\theta = 0$, and when $x = a$, $\theta = \pi/4$.

$$\therefore\ I = \int_0^{\pi/4}\frac{a\sec^2\theta\,d\theta}{(a^2 + a^2\tan^2\theta)^2} = \frac{1}{a^3}\int_0^{\pi/4}\frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$= \frac{1}{a^3}\int_0^{\pi/4}\cos^2\theta\,d\theta = \frac{1}{2a^3}\int_0^{\pi/4}(1 + \cos 2\theta)d\theta$$

$$= \frac{1}{2a^3}\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi/4} = \frac{1}{2a^3}\Big[\frac{\pi}{4} + \frac{1}{2}\Big]$$

$$= \frac{\pi + 2}{8a^3}.$$

Example 17.—Evaluate $I = \int_1^0 \sqrt{\frac{x}{1 - x}}\,dx$.

Let $\theta = \sin^{-1}\sqrt{x}$, i.e. $x = \sin^2\theta$, $\therefore\ dx = 2\sin\theta\cos\theta\,d\theta$.

When $x = 0$, $\theta = 0$, and when $x = 1$, $\theta = \pi/2$.

$$\therefore\ I = \int_{\pi/2}^{0} \frac{\sin\theta \,.\, 2\sin\theta\cos\theta\, d\theta}{\cos\theta} = \int_{\pi/2}^{0} 2\sin^2\theta\, d\theta$$

$$= \int_{\pi/2}^{0} (1 - \cos 2\theta)\, d\theta = \Big[\theta - \tfrac{1}{2}\sin 2\theta\Big]_{\pi/2}^{0}$$

$$= -\pi/2.$$

Example 18.—Find $I = \int \frac{x^{\frac{1}{2}}}{x - 1}\, dx$.

Let $\qquad x^{\frac{1}{2}} = z, \quad \therefore\ x = z^2$ and $dx = 2z\, dz$.

$$\therefore\ I = \int \frac{z \,.\, 2z}{z^2 - 1}\, dz = 2\int \frac{(z^2 - 1) + 1}{z^2 - 1}\, dz$$

$$= 2\left\{\int dz + \int \frac{dz}{z^2 - 1}\right\}$$

$$= 2\left\{z + \tfrac{1}{2}\log_e \frac{z - 1}{z + 1}\right\} + C$$

$$= 2\left\{x^{\frac{1}{2}} + \tfrac{1}{2}\log_e \frac{x^{\frac{1}{2}} - 1}{x^{\frac{1}{2}} + 1}\right\} + C.$$

N.B.—This result is obtained on the assumption that $x^{\frac{1}{2}} > 1$. If $x^{\frac{1}{2}} < 1$, the result would be

$$2\left\{x^{\frac{1}{2}} - \tfrac{1}{2}\log_e \frac{1 + x^{\frac{1}{2}}}{1 - x^{\frac{1}{2}}}\right\} + C.$$

11. Simplification before integration.

As in differentiation, it is sometimes necessary to simplify the integrand before integrating.

If the integrand consists of $\sqrt{(a_1x + b_1)}/\sqrt{(a_2x + b_2)}$, the simplification is done by multiplying the numerator and denominator by $\sqrt{(a_1x + b_1)}$, and if the integrand has a surdic denominator it is done by rationalizing the denominator.

Example 19.—Evaluate $I = \int_0^2 \sqrt{\frac{2 + x}{2 - x}}\, dx$.

$$I = \int_0^2 \frac{2 + x}{\sqrt{(4 - x^2)}} = \int_0^2 \frac{2}{\sqrt{(4 - x^2)}} - \int_0^2 \frac{-x}{\sqrt{(4 - x^2)}}\, dx$$

$$= \Big[2\sin^{-1}\frac{x}{2}\Big]_0^2 - \Big[\sqrt{(4 - x^2)}\Big]_0^2$$

$$= 2 \times \frac{\pi}{2} - [0 - 2] = 2 + \pi.$$

Note.—Example 17 can be done in a similar manner.

Example 20.—Find $I = \int \frac{dx}{\sqrt{(ax+b)} - \sqrt{(ax+c)}}$.

$$I = \int \frac{\sqrt{(ax+b)} + \sqrt{(ax+c)}}{(ax+b)-(ax+c)}\,dx = \frac{1}{b-c}\int \{(ax+b)^{\frac{1}{2}} + (ax+c)^{\frac{1}{2}}\}\,dx$$

$$= \frac{1}{a(b-c)}\left[\frac{(ax+b)^{3/2}}{3/2} + \frac{(ax+c)^{3/2}}{3/2}\right] + C$$

$$= \frac{2}{3a(b-c)}[(ax+b)^{3/2} + (ax+c)^{3/2}] + C.$$

12. Integration by parts (integration of a product).

If u and w are both functions of x, then

$$\frac{d}{dx}(uw) = u\frac{dw}{dx} + w\frac{du}{dx},$$

i.e.
$$u\frac{dw}{dx} = \frac{d}{dx}(uw) - w\frac{du}{dx}.$$

Integrating this with respect to x,

$$\int\left(u\frac{dw}{dx}\right)dx = uw - \int\left(w\frac{du}{dx}\right)dx. \quad . \quad . \quad . \quad (3)$$

Let
$$\frac{dw}{dx} = v, \quad \therefore w = \int v\,dx,$$

and, using this in (3), it becomes

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx.$$

This result is known as the formula for **integration by parts**, and, stated in words, is as follows:

"The integral of the product of two functions is equal to the product of the first and the integral of the second diminished by the integral of the product of the differential coefficient of the first and the integral of the second."

Note.—(1) When limits are involved it is safest to insert the limits only when the complete integration has been made in the two separate parts.

(2) Integration by parts is useful in finding integrals of single

functions which cannot be obtained by other means. In these cases the integrand is chosen as the first function (u), and unity as the second function (v).

Example 21.—Find $I = \int \log_e x\, dx$, and evaluate $I = \int_0^1 \sin^{-1} x\, dx$.

$$\int \log_e x \,.\, 1\, dx = \log_e x \int 1\, dx - \int \left\{\frac{d}{dx}(\log_e x) \int 1\, dx\right\} dx$$

$$= x \log_e x - \int \left(\frac{1}{x} \times x\right) dx + C$$

$$= x \log_e x - x + C.$$

$$\int_0^1 \sin^{-1} x \,.\, 1\, dx = \left[\sin^{-1} x \int 1\, dx\right]_0^1 - \int_0^1 \left[\left(\frac{d}{dx}\sin^{-1} x\right) \int 1\, dx\right] dx$$

$$= \left[x \sin^{-1} x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{(1 - x^2)}}\, dx$$

$$= \left[\frac{\pi}{2}\right] + \left[\sqrt{(1 - x^2)}\right]_0^1 = \frac{\pi}{2} - 1.$$

(3) In cases where one of the two functions u and v cannot be integrated by ordinary means, this function is chosen as the u function, and the other (which can be integrated) as the v function.

Example 22.—Find the indefinite integral $I = \int x \tan^{-1} x\, dx$.

Using $u = \tan^{-1} x$, and $v = x$,

$$I = \tan^{-1} x \int x\, dx - \int \left\{\frac{d}{dx}(\tan^{-1} x) \int x\, dx\right\} dx.$$

$$= \frac{x^2}{2} \tan^{-1} x - \tfrac{1}{2} \int \frac{x^2}{1 + x^2}\, dx + C$$

$$= \frac{x^2}{2} \tan^{-1} x - \tfrac{1}{2} \int \frac{(1 + x^2) - 1}{1 + x^2}\, dx + C$$

$$= \frac{x^2}{2} \tan^{-1} x - \tfrac{1}{2} \int 1\, dx + \tfrac{1}{2} \int \frac{1}{1 + x^2}\, dx + C$$

$$= \frac{x^2}{2} \tan^{-1} x - \tfrac{1}{2} x + \tfrac{1}{2} \tan^{-1} x + C.$$

(4) In cases when both the u and v functions are integrable by normal methods and one of the functions is x^n, where n is a positive integer, then this function is chosen as the u function.

Example 23.—Find $I = \int x^2 \cos x \, dx$.

$$I = x^2 \int \cos x \, dx - \int \left[\frac{d}{dx}(x^2) \int \cos x \right] dx$$

$$= x^2 \sin x - 2 \int x \sin x \, dx$$

$$= x^2 \sin x - 2\left[x \int \sin x \, dx - \int \left\{ \frac{d}{dx}(x) \int \sin x \, dx \right\} dx \right]$$

$$= x^2 \sin x - 2\left[-x \cos x + \int \cos x \, dx \right]$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

It is to be noted that, in this example, the integration by parts theorem has to be applied twice, and, in general, with x^n as the u function, it has to be applied n times successively.

(5) In cases where the integrand involves powers of $\cos x$ or $\sin x$, it is necessary first to convert these into linear expressions in terms of the sines and cosines of the multiple angles.

Example 24.—Find $I = \int x \cos^2 x \, dx$.

$$I = \tfrac{1}{2} \int x(1 + \cos 2x) \, dx$$

$$= \tfrac{1}{2} \int x \, dx + \tfrac{1}{2} \int x \cos 2x \, dx$$

$$= \tfrac{1}{4} x^2 + \tfrac{1}{2} \left[\frac{x}{2} \sin 2x - \int \frac{\sin 2x}{2} \, dx \right]$$

$$= \tfrac{1}{4} x^2 + \tfrac{1}{4} x \sin 2x + \tfrac{1}{8} \cos 2x + C.$$

(6) When dealing with integration by parts, the original integral sometimes occurs in the result, and can thence be found by algebraic methods as shown in the following theorem:

Theorem.—Find the indefinite integrals

$$I_1 = \int e^{kt} \sin pt \, dt \text{ and } I_2 = \int e^{kt} \cos pt \, dt,$$

where p and k are constants.

Integrating by parts,

$$I_1 = \sin pt\left(\frac{1}{k}e^{kt}\right) - \frac{p}{k}\int e^{kt}\cos pt\,dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$= \frac{1}{k}e^{kt}\sin pt - \frac{p}{k}\left[\frac{1}{k}e^{kt}\cos pt + \frac{p}{k}\int e^{kt}\sin pt\,dt\right]$$

$$= \frac{1}{k}e^{kt}\sin pt - \frac{p}{k^2}e^{kt}\cos pt - \frac{p^2}{k^2}I_1.$$

$$\therefore \quad I_1\left(1 + \frac{p^2}{k^2}\right) = \frac{1}{k^2}e^{kt}[k\sin pt - p\cos pt].$$

$$\therefore \quad I_1 = \frac{e^{kt}}{p^2 + k^2}[k\sin pt - p\cos pt].$$

Using equation (a),

$$I_1 = \frac{1}{k}e^{kt}\sin pt - \frac{p}{k}I_2.$$

$$\therefore \frac{p}{k}I_2 = \frac{1}{k}e^{kt}\sin pt - \frac{e^{kt}}{p^2 + k^2}[k\sin pt - p\cos pt]$$

$$= \frac{e^{kt}}{k(p^2 + k^2)}[(p^2 + k^2)\sin pt - k^2\sin pt + pk\cos pt].$$

$$\therefore I_2 = \frac{e^{kt}}{p^2 + k^2}[p\sin pt + k\cos pt].$$

Using $p = r\sin\alpha$, $k = r\cos\alpha$, where r is positive and $0 < \alpha < 360°$, then,

$$r = \sqrt{(p^2 + k^2)}, \quad \tan\alpha = p/k,$$

and the results for I_1 and I_2 become

$$I_1 = \frac{1}{r}e^{kt}\sin(pt - \alpha),$$

$$I_2 = \frac{1}{r}e^{kt}\cos(pt - \alpha).$$

13. Reduction formulæ.

If an integral contain constants m, n, etc., and the integral can be obtained in terms of a function containing values of $(m - 1)$, $(m - 2)$, $(n - 1)$, etc., the result is known as a *reduction formula*.

Thus, if $I_m = \int f(x, m)dx$, where m is a constant, and I_m can be

found in terms of $I_{m-1} = \int f(x, m-1)dx$, and/or I_{m-2}, etc., the result is known as a reduction formula for I_m.

The most important of the reduction formulæ, which are generally obtained by *integration by parts*, are given in the following theorem:

Theorem.—If m and n be positive integers, and

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx,$$

then
$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n} \qquad (m \geqslant 2, n \geqslant 0)$$

and
$$I_{m,n} = \frac{n-1}{m+n} I_{m,n-2} \qquad (n \geqslant 2, m \geqslant 0).$$

$$I_{m,n} = \int_0^{\pi/2} \sin^{m-1} x (\cos^n x \sin x)\, dx$$

$$= \left[\sin^{m-1} x\left(-\frac{\cos^{n+1} x}{n+1}\right)\right]_0^{\pi/2} + \frac{m-1}{n+1}\int_0^{\pi/2} (\sin^{m-2} x \cos x)\cos^{n+1} x\, dx$$

$$= 0 + \frac{m-1}{n+1}\int_0^{\pi/2} \sin^{m-2} x \cos^n x (1 - \sin^2 x)\, dx$$

$$= \frac{m-1}{n+1}\left[\int_0^{\pi/2} \sin^{m-2} x \cos^n x\, dx - \int_0^{\pi/2} \sin^m x \cos^n x\, dx\right]$$

$$= \frac{m-1}{n+1}[I_{m-2,n} - I_{m,n}].$$

$$\therefore\ I_{m,n}[(n+1) + (m-1)] = (m-1)I_{m-2,n},$$

$$\therefore\ I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}.$$

Also $I_{m,n}$

$$= \int_0^{\pi/2} (\sin^m x \cos x)\cos^{n-1} x\, dx$$

$$= \left[\frac{1}{m+1}\sin^{m+1} x \cos^{n-1} x\right]_0^{\pi/2} + \frac{n-1}{m+1}\int_0^{\pi/2} \sin^{m+1} x(\cos^{n-2} x \sin x)dx$$

$$= 0 + \frac{n-1}{m+1}\int_0^{\pi/2} \sin^m x \cos^{n-2} x(1 - \cos^2 x)\, dx$$

$$= \frac{n-1}{m+1}[I_{m,n-2} - I_{m,n}].$$

$$\therefore\ I_{m,n}(m+1+n-1) = (n-1)I_{m,n-2},$$

$$\therefore\ I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}.$$

Note.—When these two formulæ are employed it is necessary that they should be quoted, and, by continued use of them, it is possible to reduce the given integral to one of the following:

$$\int_0^{\pi/2} dx,\ \int_0^{\pi/2} \cos x\, dx, \int_0^{\pi/2} \sin x\, dx,\ \int_0^{\pi/2} \sin x \cos x\, dx,$$

each of which can be evaluated.

Example 25.—Evaluate (i) $I = \int_0^{\pi/2} \sin^4 x \cos^3 x\, dx$, (ii) $I = \int_0^{\pi/2} \cos^6 x\, dx$.

(i) By continued use of the reduction formulæ of the previous theorem,

$$I = \frac{4-1}{4+3}\int_0^{\pi/2} \sin^2 x \cos^3 x\, dx = \frac{3}{7} \times \frac{3-1}{2+3}\int_0^{\pi/2} \sin^2 x \cos x\, dx$$

$$= \frac{3}{7} \times \frac{2}{5} \times \frac{2-1}{2+1}\int_0^{\pi/2} \cos x\, dx = \frac{3}{7} \times \frac{2}{5} \times \frac{1}{3}\Big[\sin x\Big]_0^{\pi/2}$$

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(ii) Using the reduction formulæ obtained,

$$I = \frac{5}{6}\int_0^{\pi/2} \cos^4 x\, dx = \frac{5}{6} \times \frac{3}{4}\int_0^{\pi/2} \cos^2 x\, dx$$

$$= \frac{5}{8} \times \frac{1}{2}\int_0^{\pi/2} dx = \frac{5}{16}\Big[x\Big]_0^{\pi/2} \quad \frac{5}{16} \times \frac{\pi}{2} = \frac{5\pi}{32}.$$

N.B.—When deriving reduction formulæ in particular problems, clues as to procedure can generally be obtained from the result required, as in all mathematical problems. When the integrand consists of the product of two functions, neither of which can be integrated immediately, it must be manipulated into the product of two other functions, one of which (at least) can be immediately integrated, as in the previous theorem.

Example 26 (L.U.).—If $I_n = \int_0^{\pi/2} x^n \sin(2p+1)x \,.\, dx$, prove that

$$I_n + \frac{n(n-1)}{(2p+1)^2} I_{n-2} = (-1)^p \frac{n}{(2p+1)^2}\left(\frac{\pi}{2}\right)^{n-1},$$

n and p being positive integers.

Evaluate $\int_0^{\pi/2} x^3 \sin 3x\, dx$.

$$I_n = \int_0^{\pi/2} x^n \sin(2p+1)x\, dx$$

$$= \left[x^n\left\{\frac{-\cos(2p+1)x}{2p+1}\right\}\right]_0^{\pi/2} + \frac{n}{2p+1}\int_0^{\pi/2} x^{n-1}\cos(2p+1)x\, dx$$

(integration by parts)

$$= -\left(\frac{\pi}{2}\right)^n \frac{\cos(2p+1)\pi/2}{2p+1}$$

$$+ \frac{n}{2p+1}\left\{\left[x^{n-1}\frac{\sin(2p+1)x}{2p+1}\right]_0^{\pi/2} - \frac{n-1}{2p+1}\int_0^{\pi/2} x^{n-2}\sin(2p+1)x\, dx\right\}$$

$$= 0 + \frac{n}{2p+1}\left\{\left(\frac{\pi}{2}\right)^{n-1}\frac{\sin(2p+1)\pi/2}{2p+1} - \frac{n-1}{2p+1} I_{n-2}\right\}$$

$$= (-1)^p \frac{n}{(2p+1)^2}\left(\frac{\pi}{2}\right)^{n-1} - \frac{n(n-1)}{(2p+1)^2} I_{n-2}.$$

$$\therefore\ I_n + \frac{n(n-1)}{(2p+1)^2} I_{n-2} = (-1)^p \frac{n}{(2p+1)^2}\left(\frac{\pi}{2}\right)^{n-1}.$$

If $p = 1$, then $I_n = \int_0^{\pi/2} x^n \sin 3x\, dx$, and the reduction formula becomes

$$I_n = -\frac{n}{9}\left(\frac{\pi}{2}\right)^{n-1} - \frac{n(n-1)}{9} I_{n-2}.$$

Using $n = 3$ in this, $\qquad I_3 = -\frac{1}{3}\frac{\pi^2}{4} - \frac{2}{3} I_1.$

$$I_1 = \int_0^{\pi/2} x \sin 3x\, dx = \left[-\frac{x}{3}\cos 3x + \frac{1}{3}\int \cos 3x\, dx\right]_0^{\pi/2}$$

$$= \left[-\frac{x}{3}\cos 3x + \frac{1}{9}\sin 3x\right]_0^{\pi/2} = -\frac{1}{9}.$$

$$\therefore\ I_3 = \int_0^{\pi/2} x^3 \sin 3x\, dx = -\frac{\pi^2}{12} - \frac{2}{3}\left(-\frac{1}{9}\right)$$

$$= \frac{2}{27} - \frac{\pi^2}{12}.$$

Example 27 (L.U.).—If $I_n = \int \frac{t^n\, dt}{1+t^2}$, show that

$$I_{n+2} = \frac{t^{n+1}}{n+1} - I_n.$$

Evaluate $\int_0^1 \frac{t^6}{(1+t^2)}\,dt$.

$$I_{n+2} = \int \frac{t^{n+2}}{1+t^2}\,dt = \int \frac{t^n\{(1+t^2)-1\}}{1+t^2}\,dt$$

$$= \int \left(t^n - \frac{t^n}{1+t^2}\right)dt \qquad \left[\text{require } \frac{t^{n+1}}{n+1} \text{ in result}\right]$$

$$= \frac{t^{n+1}}{n+1} - I_n.$$

Now $$I_0 = \int \frac{dt}{1+t^2} = \tan^{-1} t.$$

Using the reduction formula,

for $n = 0$, $$I_2 = t - I_0 = t - \tan^{-1} t;$$

$n = 2$, $$I_4 = \frac{t^3}{3} - I_2 = \frac{t^3}{3} - t + \tan^{-1} t;$$

$n = 4$, $$I_6 = \frac{t^5}{5} - I_4 = \frac{t^5}{5} - \frac{t^3}{3} + t - \tan^{-1} t.$$

$$\therefore \int_0^1 \frac{t^6}{(1+t^2)}\,dt = \left[\frac{t^5}{5} - \frac{t^3}{3} + t - \tan^{-1} t\right]_0^1$$

$$= \left(\frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}\right) - 0$$

$$= \frac{13}{15} - \frac{\pi}{4}.$$

Note.—This result could also be obtained by using partial fractions.

Example 28 (L.U.).—If $I_n = \int x^n(a-x)^{\frac{1}{2}}\,dx$, prove that

$$(2n+3)I_n = 2anI_{n-1} - 2x^n(a-x)^{3/2}.$$

Evaluate $\int_0^a x^2(ax - x^2)^{\frac{1}{2}}\,dx$.

Integrating by parts,

$$I_n = x^n \frac{(a-x)^{3/2}}{-3/2} - \int nx^{n-1} \frac{(a-x)^{3/2}}{-3/2}\,dx$$

$$= -\tfrac{2}{3}x^n(a-x)^{3/2} + \tfrac{2}{3}n\int x^{n-1}(a-x)(a-x)^{1/2}\,dx.$$

$$\therefore\ 3I_n = -2x^n(a-x)^{3/2} + 2n\left\{\int ax^{n-1}(a-x)^{1/2}\,dx - \int x^n(a-x)^{1/2}\,dx\right\}$$

$$= -2x^n(a-x)^{3/2} + 2anI_{n-1} - 2nI_n.$$

$$\therefore\ (2n+3)I_n = 2anI_{n-1} - 2x^n(a-x)^{3/2}.$$

Using I'_n to represent $\int_0^a x^n(a-x)^{1/2}\,dx$ (n positive), the above result becomes

$$(2n+3)I'_n = 2anI'_{n-1} - \Big[2x^n(a-x)^{3/2}\Big]_0^a$$
$$= 2anI'_{n-1}.$$
$$\therefore\ I'_n = \frac{2an}{2n+3}I'_{n-1}. \qquad \text{(i)}$$

Now $\qquad \int_0^a x^2(ax-x^2)^{1/2}\,dx = \int_0^a x^{5/2}(a-x)^{1/2}\,dx = I'_{5/2}.$

But $\qquad I'_{1/2} = \int_0^a x^{\frac{1}{2}}(a-x)^{1/2}\,dx = \int_0^a (ax-x^2)^{1/2}\,dx$

$$= \int_0^a \left\{\frac{a^2}{4} - (x-a/2)^2\right\}^{1/2} dx.$$

Let $\qquad x - a/2 = a/2 \,.\, \sin\theta, \quad \therefore\ dx = a/2 \,.\, \cos\theta\,d\theta.$

When $x = 0$, $\theta = -\pi/2$, and when $x = a$, $\theta = \pi/2$.

$$\therefore\ I'_{1/2} = \int_{-\pi/2}^{\pi/2}\left\{\frac{a^2}{4} - \frac{a^2}{4}\sin^2\theta\right\}^{1/2}\frac{a}{2}\cos\theta\,d\theta$$
$$= \frac{a^2}{4}\int_{-\pi/2}^{+\pi/2}\cos^2\theta\,d\theta = \frac{a^2}{8}\int_{-\pi/2}^{+\pi/2}(1+\cos 2\theta)\,d\theta$$
$$= \frac{a^2}{8}\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_{-\pi/2}^{+\pi/2} = \frac{a^2}{8}\left[\frac{\pi}{2} - \left(\frac{-\pi}{2}\right)\right]$$
$$= \pi a^2/8.$$

Using the result (i),

for $n = 3/2$, $\qquad I'_{3/2} = \dfrac{3a}{6}I'_{1/2} = \dfrac{a}{2}\times\dfrac{\pi a^2}{8} = \dfrac{\pi a^3}{16};$

$n = 5/2$, $\qquad I'_{5/2} = \dfrac{5a}{8}\,.\,I'_{3/2} = \dfrac{5a}{8}\times\dfrac{\pi a^3}{16} = \dfrac{5\pi a^4}{128}.$

$$\therefore\ \int_0^a x^2(ax-x^2)^{1/2}\,dx = \frac{5\pi a^4}{128}.$$

Example 29 (L.U.).—(i) Prove, without integrating, that

$$\int_0^{\pi/2}\frac{dx}{1+\sin x} = \int_0^{\pi/2}\frac{dx}{1+\cos x},$$

and then obtain the value of each integral.

(ii) Evaluate (*a*) $\displaystyle\int_0^1 \frac{dx}{(x+1)(x^2+1)}$, $\qquad$ (*b*) $\displaystyle\int_0^\pi x\cos 2x\sin x\,dx$.

(i) In this type of problem it is advisable to introduce a new variable z, i.e. use the method of substitution.

Let
$$I = \int_0^{\pi/2} \frac{dx}{1 + \sin x},$$
and
$$x = \frac{\pi}{2} - z. \quad \text{Then } dx = -dz.$$

When $x = 0$, $z = \frac{\pi}{2}$, and when $x = \frac{\pi}{2}$, $z = 0$.

$$\therefore \quad I = \int_{\pi/2}^{0} - \frac{dz}{1 + \sin(\pi/2 - z)} = -\int_{\pi/2}^{0} \frac{dz}{1 + \cos z}$$
$$= \int_0^{\pi/2} \frac{dz}{1 + \cos z} \qquad \left(-\int_a^b = \int_b^a\right).$$
$$\therefore \quad \int_0^{\pi/2} \frac{dx}{1 + \sin x} = \int_0^{\pi/2} \frac{dx}{1 + \cos x} \qquad \text{(replacing } z \text{ by } x\text{)}.$$

Now
$$I = \int_0^{\pi/2} \frac{dx}{1 + \cos x} = \int_0^{\pi/2} \frac{dx}{2\cos^2(x/2)} = \int_0^{\pi/2} \tfrac{1}{2}\sec^2(x/2)\, dx$$
$$= \left[\tan\frac{x}{2}\right]_0^{\pi/2} = \left[\tan\frac{\pi}{4} - \tan 0\right] = 1.$$

(ii) (*a*) Let
$$I = \int_0^1 \frac{dx}{(x + 1)(x^2 + 1)}.$$

Now
$$\frac{1}{(x + 1)(x^2 + 1)} \equiv \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1},$$
$$\therefore \quad 1 \equiv A(x^2 + 1) + (Bx + C)(x + 1).$$

Using, in this identity,

$x = -1$, $\qquad 1 = 2A, \quad \therefore A = \tfrac{1}{2}$;

$x = 0$, $\qquad 1 = A + C, \quad \therefore C = \tfrac{1}{2}$;

$x = 1$, $\qquad 1 = 2A + 2B + 2C$
$$= 1 + 2B + 1.$$
$$\therefore B = -\tfrac{1}{2}.$$

Hence
$$I = \tfrac{1}{2}\int_0^1 \left(\frac{1}{x + 1} - \frac{x - 1}{x^2 + 1}\right) dx$$
$$= \tfrac{1}{2}\int_0^1 \left(\frac{1}{x + 1} - \frac{x}{x^2 + 1} + \frac{1}{x^2 + 1}\right) dx$$
$$= \tfrac{1}{2}\left[\log_e(x + 1) - \tfrac{1}{2}\log_e(x^2 + 1) + \tan^{-1} x\right]_0^1$$
$$= \tfrac{1}{2}\left[\left(\log_e 2 - \tfrac{1}{2}\log_e 2 + \frac{\pi}{4}\right) - 0\right]$$
$$= \tfrac{1}{2}\left[\tfrac{1}{2}\log_e 2 + \frac{\pi}{4}\right] = \tfrac{1}{4}\log_e 2 + \frac{\pi}{8}.$$

(b) $\int_0^\pi x\cos 2x\sin x\,dx = \frac{1}{2}\int_0^\pi x(\sin 3x - \sin x)\,dx$

$$= \frac{1}{2}\left\{\left[x\frac{(-\cos 3x)}{3}\right]_0^\pi + \frac{1}{3}\int_0^\pi \cos 3x\,dx\right\} - \frac{1}{2}\left\{\left[x(-\cos x)\right]_0^\pi + \int_0^\pi \cos x\,dx\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{3}\pi + \frac{1}{9}\left[\sin 3x\right]_0^\pi\right\} - \frac{1}{2}\left\{\pi + \left[\sin x\right]_0^\pi\right\}$$

$$= \frac{1}{6}\pi - \frac{1}{2}\pi = -\frac{\pi}{3}.$$

EXAMPLES ON CHAPTER V

Find the following integrals:

1. $\int \frac{ds}{\sqrt[3]{(2s^2)}}$.

2. $\int (2\sqrt{y} - 5y^{1\cdot 2} + 2y^{-0\cdot 2} + y^2 + 3)\,dy$.

3. $\int \frac{3}{(2-3z)^3}\,dz$.

4. $\int (2x+5)^4\,dx$.

5. $\int_0^4 \frac{dt}{\sqrt{(2t+1)}}$.

6. $\int \frac{2dp}{1-3p}$.

7. $\int 2e^{3x+2}\,dx$.

8. $\int \frac{1}{5e^{0\cdot 3y}}\,dy$.

9. $\int 2^{3z}\,dz$.

10. $\int \sin(2-3\theta)\,d\theta$.

11. $\int_0^\pi \cos\frac{1}{3}\phi\,.\,d\phi$.

12. $\int_0^3 \frac{dt}{9+t^2}$.

13. $\int_0^2 \frac{dx}{16-x^2}$.

14. $\int_0^1 \frac{dy}{\sqrt{(2y-y^2)}}$.

15. $\int \frac{dz}{\sqrt{(9+4z^2)}}$.

16. $\int \frac{ds}{\sqrt{(9s^2-1)}}$.

17. $\int \frac{4t+1}{4t^2+2t-1}\,dt$.

18. $\int \frac{\sinh x + \cos x}{\sin x + \cosh x}\,dx$.

19. $\int \frac{2p^2-1}{\sqrt{(2p^3-3p+2)}}\,dp$.

20. $\int \frac{x^2}{(x+1)(x-2)}\,dx$.

21. $\int_0^1 \frac{x^4}{1+x^2}\,dx$.

22. $\int \frac{dz}{z^2+2z+10}$.

23. $\int \frac{dy}{\sqrt{(y^2+2y+10)}}$.

24. $\int_0^2 \frac{3+2s}{1+2s}\,ds$.

25. $\int \frac{p+5}{p^2+4p+5}\,dp$.

26. $\int \frac{u+5}{\sqrt{(u^2+4u+5)}}\,du$.

27. $\int \frac{dx}{x^2(x^2+a^2)}$.

28. $\int_0^2 \frac{dz}{z^3+8}$.

29. $\int \frac{y^3}{(a^2+y^2)^2}\,dy$.

30. $\int \frac{x-3}{\sqrt{(4x^2-4x-3)}}$.

31. $\int \frac{2p+5}{\sqrt{(18p-9p^2-5)}}\,dp$.

32. $\int \frac{dz}{(z^2-1)(z^2-3z+2)}$.

33. $\int \frac{du}{u^4+3u^2+2}$.

34. $\int \sqrt{(10+6x+x^2)}\,dx$.

35. $\int \sqrt{(4v^2-8v-5)}\,dv$.

36. $\int_0^{2\pi} \sin 4\theta\,d\theta$.

37. $\int_0^{\pi/8} \frac{d\phi}{\cos^2 2\phi}$.

38. $\int \sin^2\frac{1}{2}x\cos\frac{1}{2}x\,dx$.

39. $\int \frac{du}{3\sin u + 4\cos u}$.

40. $\int \frac{dt}{(5\sin t - 12\cos t)^2}$.

41. $\int_0^{\pi/2} \frac{d\theta}{4+5\cos^2\theta}$.

42. $\int \frac{dv}{4+5\cos v}$.

43. $\int_0^{\pi/3} \sin^3 2\theta\, d\theta$.

44. $\int \cos 2n\theta \sin 3n\theta\, d\theta$.

45. $\int \cos 2nt \cos 5nt\, dt$.

46. $\int \sin 5\phi \sin 7\phi\, d\phi$.

47. $\int_0^{\pi/2} \sin^6 x\, dx$.

48. $\int_0^{\pi/2} \sin^3 x \cos^4 x\, dx$.

49. $\int \sin^6 t\, dt$.

50. $\int \sin^2\theta \cos^4\theta\, d\theta$.

51. $\int_{-\pi/2}^{\pi/2} \sin^5\phi \cos^3\phi\, d\phi$.

52. $\int_0^{\pi/4} \tan^4 x\, dx$.

53. $\int \tanh^3 v\, dv$.

54. $\int \sinh^2 t \cosh^2 t\, dt$.

55. $\int \operatorname{cosec}^4\theta\, d\theta$.

56. $\int \cosh^2 2\phi\, d\phi$.

57. $\int \frac{dy}{\sqrt{(y^2+1)}-y}$.

58. $\int \sqrt{\left(\frac{3-x}{x-1}\right)}\, dx$.

59. $\int \frac{z\, dz}{(a^2-z^2)^{5/2}}$.

60. $\int \frac{dp}{(a^2+p^2)^{3/2}}$.

61. $\int_0^a (a^2-v^2)^{3/2}\, dv$.

62. $\int \frac{x^2}{\sqrt{(1-3x)}}\, dx$.

63. $\int_0^{2a} \frac{x}{\sqrt{(2ax-x^2)}}\, dx$.

64. $\int_a^b \frac{dy}{\sqrt{\{(y-a)(b-y)\}}}$.

65. $\int_0^1 x^3(1-x^2)^{3/2}\, dx$.

66. $\int \frac{\cos\theta}{1+2\cos\theta}\, d\theta$.

67. $\int \frac{1}{x\sqrt{(x^2-1)}}\, dx$.

68. $\int y \cosh y\, dy$.

69. $\int \theta \sin^2 2\theta\, d\theta$.

70. $\int \sin^{-1} x\, dx$.

71. $\int v^3 \log_e v\, .\, dv$.

72. $\int e^{-2\theta} \sin 3\theta\, d\theta$

73. $\int x\sqrt{(1+x)}\, .\, dx$.

74. $\int u^2 e^{-\frac{1}{2}u}\, du$.

75. $\int_0^{\pi/2} \sin^3\theta(\sin^3\theta + \cos^3\theta)\, d\theta$.

76. $\int \sec^{-1} x\, dx$.

77. $\int x \log_e(x+1)\, dx$.

78. $\int \coth(2\theta+1)\, d\theta$.

79. $\int \theta^2 \cos 2\theta\, d\theta$.

80. $\int \frac{\sin^2\theta\cos\theta}{1-\sin\theta}\, d\theta$.

81. $\int \tanh^2 2u\, du$.

82. $\int \frac{dx}{x\sqrt{(1+4x^2)}}$.

83. $\int_0^{\pi/2} \sin x \cos^2 2x\, dx$.

84. $\int_{\pi/2}^{\pi/4} \cot^3\theta\, d\theta$.

The following examples are taken from London University examination papers.

85. (i) Find by the use of the substitution $t = x - \frac{1}{2}(a+b)$, the value of

$$\int_a^b (b-x)(x-a)\{x-\tfrac{1}{2}(a+b)\}\, dx.$$

(ii) Find the value of $\int_0^{\pi/2} x^2 \sin^2 x\, dx$.

86. Find the values of

$$\text{(i)} \int \frac{x^2+x+1}{x^2-x+1}\,dx, \quad \text{(ii)} \int \frac{x^{\frac{1}{2}}}{x-1}\,dx, \quad \text{(iii)} \int_0^\pi \frac{2-\sin\theta}{3+2\cos\theta}\,d\theta.$$

87. (i) Evaluate $\int \frac{dx}{x^2(1+x)}$.

(ii) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, and use this relation to prove that

$$\int_0^{\pi/2} \frac{x\,dx}{\sin x + \cos x} = \frac{\pi}{2\sqrt{2}} \log_e(1+\sqrt{2}).$$

88. Find $\int \frac{x^2-x+1}{x^2+x+1}\,dx$, and evaluate $\int_0^{\pi/2} x^2 \sin x\,dx$.

If $x^3 - y^2 = a^3$, express the differential coefficients of xy and $\log_e(x+y)$ with respect to x in terms of y. Hence find $\int y\,dx$.

89. Find (i) $\int \frac{2-3x}{x\sqrt{(1+x)}}\,dx$, (ii) $\int \frac{xe^x}{(1+x)^2}\,dx$, (iii) $\int_2^3 \frac{x\,dx}{(5x-6-x^2)^{\frac{1}{2}}}$.

90. Find (i) $\int e^{ax} \sin bx\,dx$, (ii) $\int e^{a\sin^{-1}x}\,dx$, (iii) $\int x^3 \tan^{-1} x\,dx$.

91. Integrate (i) $\frac{1}{1+x\sqrt{x}}$, and (ii) $\frac{1}{a+b\cos x}$, where $|a| > |b|$.

Evaluate $\int_0^a \frac{x(a^2-x^2)^{\frac{1}{2}}}{(a^2+x^2)^{\frac{1}{2}}}\,dx$, by means of the substitution $x^2 = a^2\cos 2\theta$, or by any other method.

92. Evaluate $\int e^{\sqrt{x}}\,dx$, $\int e^{\sin^{-1}x}\,dx$, and $\int \frac{d\theta}{\cos\alpha + \cos\theta}$.

Show that, if α be a positive acute angle,

$$\int_0^{\pi/2} \frac{d\theta}{\cos\alpha + \cos\theta} = \frac{1}{\sin\alpha} \log_e(\sec\alpha + \tan\alpha).$$

93. Find the following:

$$\text{(i)} \int \frac{(3x-1)\,dx}{(1-x)^2(1+x)}; \quad \text{(ii)} \int \frac{dx}{a+b\cos x}, \text{ when } a > b;$$

$$\text{(iii)} \int_0^{\pi/2} e^{-ax} \cos bx\,dx, \text{ when } a > 0.$$

94. Find the following integrals:

$$\text{(i)} \int_0^1 \sqrt{\left(\frac{x}{2-x}\right)}\,dx, \quad \text{(ii)} \int_0^{1/2} x^2 \sin^{-1} x\,dx, \quad \text{(iii)} \int \frac{d\theta}{\sin\theta - 2\sin^2\theta}.$$

95. Find the following integrals:

$$\text{(i)} \int \frac{dx}{(1+x)(1-x^2)^{1/2}}, \quad \text{(ii)} \int \frac{x+\sin x}{1+\cos x}\,dx, \quad \text{(iii)} \int_a^b \frac{(b-x)^{1/2}}{(x-a)^{1/2}}\,dx, \text{ where } b > a.$$

96. Evaluate (i) $\int \frac{a+2x}{\sqrt{(a^2-x^2)}}\,dx$, (ii) $\int \frac{dx}{3\cos x + 4\sin x}$, (iii) $\int x^3 \tan^{-1} x\,dx$.

97. Evaluate (i) $\int_0^a x^2(a^2-x^2)^{3/2}\,dx$, (ii) $\int_0^1 \frac{x^4}{(1+x^2)^2}\,dx$, (iii) $\int_0^1 x^3 e^{x^2}\,dx$.

98. Evaluate

(i) $\int \frac{a + b\cos\theta + \sin\theta}{1+\sin\theta}\,d\theta$, (ii) $\int \frac{x^3\,dx}{(x^2+4)^{5/2}}$, (iii) $\int_0^{\pi/4} \tan^5\theta\,d\theta$.

99. Evaluate

(i) $\int x^2 e^{-x}\,dx$, (ii) $\int_0^{\pi/4} \tan^6 x\,dx$, (iii) $\int x^{3/2}(1-x)^3\,dx$, (iv) $\int \frac{1+2\cos x}{(2+\cos x)^2}\,dx$.

100. Evaluate the following integrals:

(i) $\int_0^1 x^3 e^{-2x}\,dx$, (ii) $\int_0^\infty e^{-x}\sin 2x\,dx$, (iii) $\int (x^2+6x+13)^{1/2}\,dx$.

101. Evaluate (i) $\int_0^{\pi/2} \cos^6 x \sin^2 x\,dx$, (ii) $\int \frac{\sin x\cos x}{1 - e\cos x}\,dx$, (iii) $\int_0^4 x^3 e^{2x}\,dx$.

102. Evaluate the integrals:

(i) $\int_0^1 x\tan^{-1} x\,dx$, (ii) $\int_0^{\pi/2} \sin^4 x\cos^3 x\,dx$, (iii) $\int_0^\infty \frac{x+1}{(x+2)(x^2+1)}\,dx$.

103. Find the following indefinite integrals:

(i) $\int \frac{dx}{x^2+4x+5}$, (ii) $\int \frac{dx}{x^2+4x+3}$, (iii) $\int x^2\sin x\,dx$, (iv) $\int \frac{d\theta}{5+3\cos\theta}$.

104. Find the indefinite integrals:

(i) $\int \frac{\sqrt{(x-1)}}{(x-2)\sqrt{(x-2)}}\,dx$, (ii) $\int \frac{\sin x}{2+\sin x}\,dx$, (iii) $\int \theta\sin^3\theta\,d\theta$.

Evaluate to three significant figures $\int_0^1 \frac{x+1}{x^2+1}\,dx$.

105. Find the indefinite integrals:

$$\int \sqrt{\left(\frac{1-x}{1+x}\right)}\,dx, \quad \int \frac{d\theta}{5+3\cos\theta}, \quad \int \frac{du}{\cosh u}.$$

Evaluate to three significant figures $\int_0^2 \frac{dx}{x^3+8}$.

106. Find the indefinite integrals:

$$(a) \int \frac{\sin\theta}{\sqrt{(\sin^2\alpha - \sin^2\theta)}}\,d\theta, \quad (b) \int \frac{\cos\theta\,d\theta}{\sqrt{(\sin^2\alpha - \sin^2\theta)}}.$$

Evaluate to three decimal places:

$$(c) \int_0^1 (1-x)\sqrt[3]{x}\,dx, \quad (d) \int_0^{\pi/4} \sec^3\theta\,d\theta.$$

107. Evaluate the following, giving the numerical results correct to three significant figures:

$$(a) \int_0^{\pi/2} \frac{d\theta}{1+\cos^2\theta}, \qquad (b) \int_0^{\pi/2} \frac{\cos\theta}{1+\cos^2\theta}\,d\theta,$$
$$(c) \int_0^{\pi/2} \frac{\cos^2\theta}{1+\cos^2\theta}\,d\theta, \quad (d) \int_0^{\pi/2} \frac{\sin\theta}{1+\cos^2\theta}\,d\theta.$$

108. Find the values of $\int_0^{\pi/2} \sin^n x\,dx$, where n is a positive integer, distinguishing between the cases when n is odd and when n is even.

If the equation of a curve in polar co-ordinates is $r = a(\cos^4\theta + \sin^4\theta)$, find the positive values of $\theta(<\pi/2)$, for which the radius vector is a minimum. Find the area enclosed by the curve, the initial line, and this minimum radius vector (see Chap. XIV).

109. If $I_n = \int \sec^n\theta\,d\theta$, prove that $(n-1)I_n = \sec^{n-2}\theta\tan\theta + (n-2)I_{n-2}$.

Evaluate $\int_0^a x^4(a^2+x^2)^{1/2}\,dx$,

110. If $I_{m,n} = \int_0^{\pi/2} \cos^m\theta\cos n\theta\,d\theta$, where m and n are positive integers, prove that $(n-m)I_{m,n} + mI_{m-1,n+1} = 0$.

Evaluate $\int_0^{\pi/2} \cos^3\theta\cos 6\theta\,d\theta$.

111. If I_n denotes $\int_0^1 x^p(1-x^q)^n\,dx$, where p, q, n are positive, prove that $(qn+p+1)I_n = qnI_{n-1}$.

Evaluate I_n when n is a positive integer.

112. Obtain a reduction formula for $\int \sin^m\theta\,d\theta$, where m is a positive integer.

Evaluate (i) $\int_0^4 x^{1/2}(4-x)^{3/2}\,dx$, (ii) $\int_0^a \frac{x^4\,dx}{(a^2+x^2)^4}$.

113. If $I_{p,q} = \int_0^{\pi/2} \cos^p\theta\cos q\theta\,.\,d\theta$, show that $(p+q)I_{p,q} = pI_{p-1,q-1}$.

Evaluate $\int_0^{\pi/2} \cos^4\theta\cos 5\theta\,d\theta$.

114. If $u_n = \int_0^{\pi} e^{-x}\sin^n x\,dx$, show that, for $n > 1$, $(n^2+1)u_n = n(n-1)u_{n-2}$.

Evaluate $\int_{-\pi/2}^{+\pi/2} e^{-x}\cos^3 x\,dx$.

115. If $I_{m,n} = \int_0^1 (1-x^m)^n\,dx$, where m and n are positive, show that $(mn+1)I_{m,n} = mnI_{m,n-1}$.

If, in addition, n is an integer, evaluate $I_{m,n}$.

116. If $u_n = \int_0^{\pi/2} x \cos^n x \, dx$, where $n > 1$, show that

$$u_n = -\frac{1}{n^2} + \frac{n-1}{n} u_{n-2}.$$

Evaluate u_4 and u_5.

117. If $I_n = \int \frac{x^n \, dx}{\sqrt{(a^2 + x^2)}}$, show that $I_n = \frac{x^{n-1}}{n} \sqrt{(a^2 + x^2)} - \frac{(n-1)}{n} a^2 I_{n-2}$,
where $n \geqslant 2$.

Evaluate $\int_0^2 \frac{x^5 \, dx}{\sqrt{(5 + x^2)}}$.

118. If $u_m = \int x^m (a^2 - x^2)^{1/2} \, dx$, show that

$$(m + 2) u_m = -x^{m-1} (a^2 - x^2)^{3/2} + a^2 (m - 1) u_{m-2}.$$

Evaluate $\int_0^{\pi/2} \sin^{2m} \theta \cos^2 \theta \, d\theta$, where m is a positive integer.

119. If $I_n = \int_0^a \frac{x^n \, dx}{\sqrt{(3a^2 + x^2)}}$, show that $I_n = \frac{2a^n}{n} - \frac{3(n-1)}{n} a^2 I_{n-2}$.

Evaluate I_7.

120. If $I_{p,q} = \int \frac{x^p}{(1 + x^2)^q} \, dx$, show that

$$2(q-1) I_{p,q} = -\frac{x^{p-1}}{(1 + x^2)^{q-1}} + (p-1) I_{p-2, q-1}.$$

Hence, or otherwise, evaluate $\int_0^1 \frac{x^6}{(1 + x^2)^3} \, dx$.

121. If $I_n = \int_0^{2a} x^n \sqrt{(2ax - x^2)} \, dx$, prove that

$$3aI_n - 3I_{n+1} = n(I_{n+1} - 2aI_n).$$

Hence, or otherwise, show that

$$I_n = \frac{(2n+1)(2n-1) \ldots 7.5}{(n+2)(n+1) \ldots 5.4} \cdot \frac{\pi}{2} a^{n+2},$$

where n is a positive integer.

122. Prove that $\int_0^n x^2 (n - x)^p \, dx = \int_0^n x^p (n - x)^2 \, dx$,
and find the common value of the two integrals.

123. Show that $\int_0^\infty e^{-x} x^n \, dx = n \int_0^\infty x^{n-1} e^{-x} \, dx$,
and deduce that, when n is a positive integer, the value of the first integral is $n!$

Find the value of a for which $\int_0^\infty (1 - ax^2)^2 e^{-x} \, dx$ is a minimum.

124. Prove that (i) $\displaystyle\int_0^{\pi/2} \log_e (\sin x)\,dx = \int_0^{\pi/2} \log_e (\cos x)\,dx$

$$= \tfrac{1}{2}\int_0^{\pi/2} \log_e (\sin 2x)\,dx - \frac{\pi}{4}\log_e 2;$$

(ii) $\displaystyle\int_0^{\pi/2} \log_e (\sin 2x)\,dx = \tfrac{1}{2}\int_0^{\pi} \log_e (\sin x)\,dx = \int_0^{\pi/2} \log_e (\sin x)\,dx.$

Deduce that $\displaystyle\int_0^{\pi/2} \log_e (\sin x)\,dx = \frac{\pi}{2}\log_e \tfrac{1}{2}.$

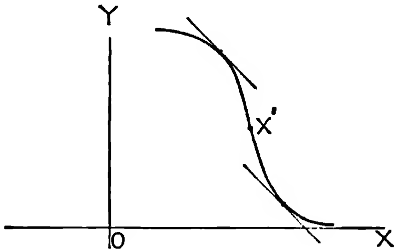
125. Obtain a reduction formula for $\displaystyle\int \cos^{2n}\theta\,d\theta$, and evaluate $\displaystyle\int_0^{r\pi/2} \cos^{2n}\theta\,d\theta$, where r is a positive integer.

upwards, and at all these points it can be shown that d^2y/dx^2 is negative as shown at A.

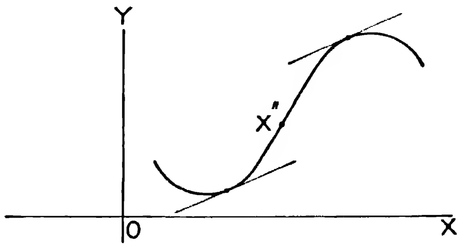
At all points in the neighbourhood of B and D the curve is *concave upwards*, and at these points, as in the case of the point B, it can be shown that d^2y/dx^2 is positive.

7. Points of Inflexion.

A *point of inflexion* on a curve is a point at which the curve changes from concave upwards to convex upwards, or vice versa (or a point on a curve at which the tangent cuts the curve in three coincident



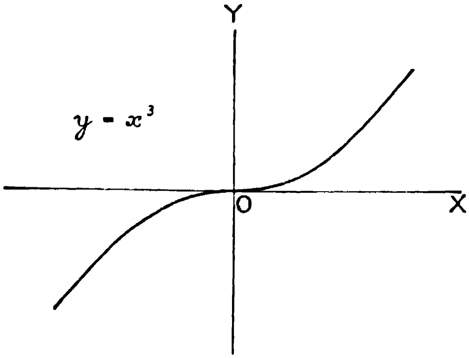
points and therefore the curve lies on both sides of the tangent). Considering the case of the change from convex upwards to concave upwards, and moving from left to right, the slope of the tangent is negative and decreasing up to the point of inflexion X′, and then negative



and increasing after the point X′. Hence from the definition of a minimum, the slope dy/dx must be a minimum at this point of inflexion, i.e. at this point of inflexion X′, $\dfrac{d^2y}{dx^2} = 0$.

Similarly, for the point of inflexion X″, where the concave upwards curve changes to convex upwards as x increases, it can be shown that dy/dx is a maximum and $d^2y/dx^2 = 0$.

It is possible, as in the case of the curve $y = x^3$ at O, to have a point of inflexion with $dy/dx = 0$. Hence $dy/dx = 0$ is not a sufficient



criterion for a turning point, and the following table gives the full conditions for turning points and points of inflexion.

Type of point	dy/dx	d^2y/dx^2
Maximum	0	Negative
Minimum	0	Positive
Point of inflexion	Any value	0

8. Theorem.—If the first $(r-1)$ derivatives of $f(x)$ vanish when $x = a$, it is required to find the nature of the point on the curve $y = f(x)$ where $x = a$, providing $f^{(r)}(a) \neq 0$.

Using Taylor's theorem with h small,

$$f(a+h) = f(a) + hf'(a) + \ldots + \frac{h^r}{r!}f^{(r)}(a) + \ldots.$$

If $f'(a), f''(a), \ldots f^{(r-1)}(a)$ vanish, this becomes

$$f(a+h) - f(a) = \frac{h^r}{r!}f^{(r)}(a),$$

neglecting other terms since h is small.

If r be an *odd* integer, then h^r changes its sign when h changes its sign. Therefore $f(a+h) - f(a)$ will change its sign as h changes its sign, if r be odd. Hence, in this case, the point given by $x = a$ is not a turning point.

If r be *even*, then h^r is always positive, no matter what the sign of h. Hence $f(a+h)-f(a)$ will retain the same sign as h changes its sign, when r is even. Thus, when r is even there is a turning point at $x=a$, which is a maximum point if $f^{(r)}(a)$ is negative, and a minimum point if $f^{(r)}(a)$ is positive.

Note.—In problems of a practical nature, the type of turning point can usually be ascertained without having recourse to the second condition, namely d^2y/dx^2 positive or negative for a minimum or maximum respectively.

Example 6 (L.U.).—Prove that the curve given by the equation

$$y^2=(x+1)(2x^2-7x+7)$$

has turning points at $x=0$ and $x=5/3$, and a point of inflexion at $x=1$. Give a sketch of the curve for values of x not greater than 3.

$$y^2=(x+1)(2x^2-7x+7). \quad \dots\dots \quad \text{(i)}$$

Differentiating (i) with respect to x,

$$2y\frac{dy}{dx}=2x^2-7x+7+(x+1)(4x-7)$$

$$=2x^2-7x+7+4x^2-3x-7$$

$$=6x^2-10x,$$

i.e.

$$y\frac{dy}{dx}=3x^2-5x. \quad \dots\dots \quad \text{(ii)}$$

Differentiating (ii) with respect to x,

$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=6x-5.$$

$$\therefore\ y\frac{d^2y}{dx^2}=6x-5-\frac{(3x^2-5x)^2}{y^2} \qquad \text{[using (ii)]}$$

$$=6x-5-\frac{9x^4-30x^3+25x^2}{(x+1)(2x^2-7x+7)}. \quad \dots \quad \text{(iii)}$$

Now, when $x=1$, the R.H.S. of (iii) is zero; therefore there is a point of inflexion when $x=1$, for $d^2y/dx^2=0$ when $x=1$.

Now $\qquad dy/dx=0$ when $3x^2-5x=0 \qquad$ [from (ii)],

i.e. when $\qquad x=0$ or $5/3$,

and, from (iii), $\qquad \dfrac{d^2y}{dx^2}\neq 0$ when $x=0$ or $5/3$,

$\therefore$ there are turning points when $x=0$ or $5/3$.

N.B.—When *sketching* the graph of an equation, the following are required:

(i) Points in which curve cuts OX and OY.
(ii) Maxima, minima, and points of inflexion.
(iii) Symmetry of curve.
(iv) Asymptotes (i.e. lines which curve touches at infinity).

From equation (i),

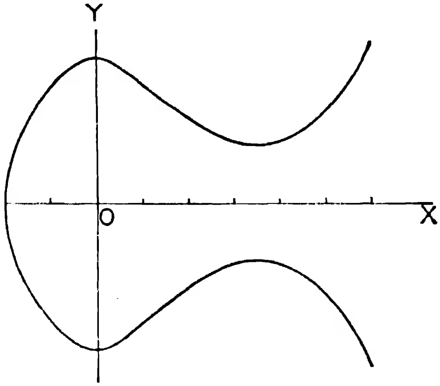
(*a*) The curve is symmetrical about OX (for every value of x there are two equal and opposite values of y);

(*b*) when $x = 0$, $y = \pm\sqrt{7}$;

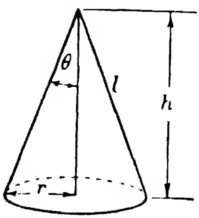
(*c*) when $y = 0$, $x = -1$ or imaginary values;

(*d*) previous results give maxima, minima, and point of inflexion.

Hence the following is a sketch of the curve.



Example 7 (L.U.).—A right circular cone, including a flat circular base, is constructed of sheet material of uniform small thickness. Express the total area of the surface in terms of the volume and vertical semi-angle θ of the cone, and show that, for a given volume, the area of the surface is a minimum if the vertical semi-angle equals $\sin^{-1}\frac{1}{3}$.



Let r = radius of base of cone,
h = its height,
l = its slant length,
V = volume of cone,
A = total surface area.

From the diagram, $h = r\cot\theta$, and $l = r\operatorname{cosec}\theta$.

Now
$$V = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi r^3 \cot\theta. \qquad \text{(i)}$$
$$A = \pi r^2 + \pi r l = \pi r^2 + \pi r^2 \operatorname{cosec}\theta$$
$$= \pi r^2(1 + \operatorname{cosec}\theta). \qquad \text{(ii)}$$

From (i), $\quad r^3 = \dfrac{3V}{\pi}\tan\theta, \quad \therefore\ r^2 = \left(\dfrac{3V}{\pi}\right)^{2/3}\tan^{2/3}\theta.$

Using this in (ii), $A = \pi\left(\dfrac{3V}{\pi}\right)^{2/3}\tan^{2/3}\theta(1 + \operatorname{cosec}\theta)$

$$= (3V)^{2/3}\pi^{1/3}\tan^{2/3}\theta(1 + \operatorname{cosec}\theta).$$

$\therefore\ A = k\tan^{2/3}\theta(1 + \operatorname{cosec}\theta)$, where $k = \pi^{1/3}(3V)^{2/3} =$ constant.

Hence $\dfrac{dA}{d\theta} = k\,\{\tan^{2/3}\theta(-\operatorname{cosec}\theta\cot\theta) + (\frac{2}{3}\tan^{-1/3}\theta\sec^2\theta)(1 + \operatorname{cosec}\theta)\}$

$$= \frac{k}{\tan^{1/3}\theta}\{\tan\theta(-\operatorname{cosec}\theta\cot\theta) + \tfrac{2}{3}\sec^2\theta(1 + \operatorname{cosec}\theta)\}$$

$$= \frac{k}{\tan^{1/3}\theta}\left\{-\frac{1}{\sin\theta} + \frac{2}{3\cos^2\theta}\left(1 + \frac{1}{\sin\theta}\right)\right\}$$

$$= \frac{k}{3\cos^2\theta\sin\theta\tan^{1/3}\theta}\{2\sin\theta + 2 - 3\cos^2\theta\}$$

$$= \frac{k}{3\cos^2\theta\sin\theta\tan^{1/3}\theta}\{3\sin^2\theta + 2\sin\theta - 1\}.$$

For a turning value of A, it follows that $\dfrac{dA}{d\theta} = 0$,

i.e. $$3\sin^2\theta + 2\sin\theta - 1 = 0,$$

i.e. $$(3\sin\theta - 1)(\sin\theta + 1) = 0.$$

$$\therefore\ \sin\theta = \tfrac{1}{3},\ \text{since}\ \sin\theta \neq -1.$$

From practical considerations it is clear that $\sin\theta = \frac{1}{3}$ gives a minimum value and not a maximum value.

Hence, for a minimum value of A,

$$\theta = \sin^{-1}\tfrac{1}{3}.$$

Example 8 (L.U.).—Find the maximum and minimum values of the function $\cos x\cos(x - \pi/6)\cos(x + \pi/6)$, where $0 \leqslant x \leqslant \pi$.

N.B.—In a problem of this nature it is advisable to express the function as the sum of certain cosines, as shown, before proceeding to differentiate.

Let $$y = \cos x\cos(x - \pi/6)\cos(x + \pi/6)$$

$$= \tfrac{1}{2}\cos x\{\cos 2x + \cos\pi/3\}$$

$$= \tfrac{1}{4}\{\cos 3x + \cos x\} + \tfrac{1}{4}\cos x$$

$$= \tfrac{1}{4}\{\cos 3x + 2\cos x\}.$$

$$\therefore\ \frac{dy}{dx} = -\tfrac{1}{4}\{3\sin 3x + 2\sin x\}. \qquad \text{(i)}$$

$$\frac{d^2y}{dx^2} = -\tfrac{1}{4}\{9\cos 3x + 2\cos x\}. \qquad \text{(ii)}$$

For turning points $dy/dx = 0$, i.e. from (i),

$$3 \sin 3x + 2 \sin x = 0,$$

i.e. $$3\{3 \sin x - 4 \sin^3 x\} + 2 \sin x = 0,$$

i.e. $$11 \sin x - 12 \sin^3 x = 0.$$

$$\therefore \ \sin x(11 - 12 \sin^2 x) = 0.$$

$$\therefore \ \sin x = 0 \ \text{ or } \ \sin x = \pm \sqrt{(11/12)}.$$

When $\sin x = 0$, $x = 0$ or π in the given range.

When $x = 0$, $y = \frac{3}{4}$ and d^2y/dx^2 is negative.

$$\therefore \ y = \tfrac{3}{4} \text{ is a maximum value.}$$

When $x = \pi$, $y = -\frac{3}{4}$ and d^2y/dx^2 is positive,

$$\therefore \ y = -\tfrac{3}{4} \text{ is a minimum value.}$$

When $\sin x = +\sqrt{(11/12)}$ ($\sin x = -\sqrt{(11/12)}$ is inadmissable since $0 \leqslant x \leqslant \pi$),

$$\cos x = \pm \frac{1}{\sqrt{12}}.$$

$$y = \tfrac{1}{4}\{\cos 3x + 2 \cos x\} = \tfrac{1}{4}\{4 \cos^3 x - 3 \cos x + 2 \cos x\}$$

$$= \tfrac{1}{4}\{4 \cos^3 x - \cos x\} = \tfrac{1}{4}\left\{\frac{1}{3\sqrt{12}} - \frac{1}{\sqrt{12}}\right\}$$

$$= -\frac{1}{2\sqrt{12}} \text{ for } + \text{ sign and } + \frac{1}{2\sqrt{12}} \text{ for } - \text{ sign.}$$

$$\frac{d^2y}{dx^2} = -\tfrac{1}{4}\{9(4 \cos^3 x - 3 \cos x) + 2 \cos x\}$$

$$= -\tfrac{1}{4}\{36 \cos^3 x - 25 \cos x\}$$

$$= -\tfrac{1}{4}\left\{\frac{3}{\sqrt{12}} - \frac{25}{\sqrt{12}}\right\} = \tfrac{1}{4}\,\frac{22}{\sqrt{12}}, \text{ which is positive for } \cos x = +1/\sqrt{12}.$$

$\therefore \ y = -\dfrac{1}{2\sqrt{12}}$ gives a minimum value of y, and $y = +\dfrac{1}{2\sqrt{12}}$ similarly gives a maximum value of y.

Hence, for $0 \leqslant x \leqslant \pi$, the expression $\cos x \cos (x - \pi/6) \cos (x + \pi/6)$ has *maximum values of* $\frac{3}{4}$ and $1/2\sqrt{12}$, *and minimum values of* $-\frac{3}{4}$ *and* $-1/2\sqrt{12}$.

9. Evaluation of Limiting values using Taylor's Theorem.

Theorem.—To find the value of $\underset{x\to a}{\text{Lt}}\dfrac{F(x)}{f(x)}$, when $F(a) = f(a) = 0$.

$$\underset{x\to a}{\text{Lt}}\frac{F(x)}{f(x)} = \underset{h\to 0}{\text{Lt}}\frac{F(a+h)}{f(a+h)}$$

$$= \underset{h\to 0}{\text{Lt}}\frac{F(a) + hF'(a) + \dfrac{h^2}{2!}F''(a) + \ldots}{f(a) + hf'(a) + \dfrac{h^2}{2!}f''(a) + \ldots}$$

$$= \underset{h\to 0}{\text{Lt}}\frac{hF'(a) + \dfrac{h^2}{2!}F''(a) + \ldots}{hf'(a) + \dfrac{h^2}{2!}f''(a) + \ldots} \qquad \text{(since } F(a) = f(a) = 0\text{)}$$

$$= \underset{h\to 0}{\text{Lt}}\frac{F'(a) + \dfrac{h}{2!}F''(a) + \ldots}{f'(a) + \dfrac{h}{2!}f''(a) + \ldots}$$

$$= \frac{F'(a)}{f'(a)},$$

providing that $F'(a)$ and $f'(a)$ are not simultaneously zero. In the case when $F'(a)$ and $f'(a)$ are simultaneously zero, it follows by a similar proof that the value of the limit is $F''(a)/f''(a)$, providing that $F''(a)$ and $f''(a)$ are not simultaneously zero.

In general, if the first $(r - 1)$ derivatives of $F(x)$ and $f(x)$ are all zero when $x = a$, the value of the given limit is $F^{(r)}(a)/f^{(r)}(a)$, where the rth derivatives of $F(x)$ and $f(x)$ are not simultaneously zero when $x = a$.

Example 9.—Evaluate $\underset{x\to a}{\text{Lt}}\dfrac{x^2\sin a - a^2\sin x}{x - a}$.

Let $F(x) = x^2\sin a - a^2\sin x$, $\quad\therefore F(a) = 0$.

Let $f(x) = x - a$, $\quad\therefore f(a) = 0$.

$F'(x) = 2x\sin a - a^2\cos x$, $\quad\therefore F'(a) = 2a\sin a - a^2\cos a$.

$f'(x) = 1$, $\quad\therefore f'(a) = 1$.

Hence
$$\underset{x\to a}{\text{Lt}}\frac{F(x)}{f(x)} = \frac{F'(a)}{f'(a)}$$
$$= 2a\sin a - a^2\cos a$$
$$= a\{2\sin a - a\cos a\}.$$

Example 10.—Evaluate $\underset{x \to 1}{\text{Lt}} \dfrac{\log_e x}{x-1}$.

Let $F(x) = \log_e x, \quad \therefore F(1) = 0.$

Let $f(x) = x - 1, \quad \therefore f(1) = 0.$

$$F'(x) = \frac{1}{x}, \quad \therefore F'(1) = 1.$$

$$f'(x) = 1, \quad \therefore f'(1) = 1.$$

$$\therefore \underset{x \to 1}{\text{Lt}} \frac{F(x)}{f(x)} = \frac{F'(1)}{f'(1)} = 1.$$

Example 11.—Evaluate $\underset{x \to 0}{\text{Lt}} \dfrac{\sin x - x}{x^3}$.

Let $F(x) = \sin x - x, \quad \therefore F(0) = 0;$ Let $f(x) = x^3, \quad \therefore f(0) = 0.$

$F'(x) = \cos x - 1, \quad \therefore F'(0) = 0; \quad f'(x) = 3x^2, \quad \therefore f'(0) = 0.$

$F''(x) = -\sin x, \quad \therefore F''(0) = 0; \quad f''(x) = 6x, \quad \therefore f''(0) = 0.$

$F'''(x) = -\cos x, \quad \therefore F'''(0) = -1; \quad f'''(x) = 6, \quad \therefore f'''(0) = 6.$

$$\therefore \underset{x \to 0}{\text{Lt}} \frac{F(x)}{f(x)} = \frac{F'''(0)}{f'''(0)} = -\frac{1}{6}.$$

Note.—This result could have been obtained by using the series for $\sin x$.

Example 12.—Evaluate $\underset{x \to \pi/2}{\text{Lt}} \dfrac{\tan 3x}{\tan x}$.

N.B.—When x is replaced by $\pi/2$ in $\tan 3x/\tan x$, the result is ∞/∞, when $0/0$ is really required, hence the given limit is transformed into the equivalent function $\underset{x \to \pi/2}{\text{Lt}} \dfrac{\cot x}{\cot 3x}$.

Let $F(x) = \cot x, \quad \therefore F(\pi/2) = 0.$

Let $f(x) = \cot 3x, \quad \therefore f(\pi/2) = 0.$

$$F'(x) = -\operatorname{cosec}^2 x, \quad \therefore F'(\pi/2) = -1.$$

$$f'(x) = -3 \operatorname{cosec}^2 3x, \quad \therefore f'(\pi/2) = -3.$$

Hence $$\underset{x \to \pi/2}{\text{Lt}} \frac{F(x)}{f(x)} = \frac{F'(\pi/2)}{f'(\pi/2)} = \frac{-1}{-3} = \frac{1}{3},$$

$$\therefore \underset{x \to \pi/2}{\text{Lt}} \frac{\tan 3x}{\tan x} = \frac{1}{3}.$$

CHAPTER VII

Tangents, Normals, Curvature, Partial Differentiation, etc.

TANGENTS AND NORMALS

1. If the equation of the curve be given in the *explicit form*, $y = f(x)$, the slope of the curve at (x, y) is dy/dx. Hence the slope of the tangent at (x_1, y_1) is $(dy/dx)_1$, where $(dy/dx)_1$ denotes the result of replacing x by x_1, and y by y_1, in dy/dx.

Thus the equation of the tangent at (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_1 (x - x_1).$$

The slope of the normal at (x_1, y_1) is $-1/(dy/dx)_1$. Hence the equation of the normal at this point after simplification will be

$$(x - x_1) + \left(\frac{dy}{dx}\right)_1 (y - y_1) = 0.$$

2. If the equation of the curve be given in the *implicit form*, $f(x, y) = 0$, dy/dx is found by differentiating the equation with respect to x, and then the procedure is the same as in Section 1.

Example 1.—Find the equation of the tangent and normal at the point (1, 1) of the ellipse $x^2 + 2y^2 = 3$.

Differentiating the given equation with respect to x,

$$2x + 4y\frac{dy}{dx} = 0, \quad \therefore \frac{dy}{dx} = -\frac{x}{2y}.$$

Therefore, at the point (1, 1) the slope of the tangent is $-\frac{1}{2}$, and the slope of the normal will be $+2$.

Hence the equation of the tangent at (1, 1) is

$$y - 1 = -\tfrac{1}{2}(x - 1),$$

i.e. $$x + 2y = 3.$$

And the equation of the normal is

$$y - 1 = 2(x - 1),$$

i.e.
$$y = 2x - 1.$$

3. If the curve be given in the *parametric form*, $x = f_1(t)$, $y = f_2(t)$, using the fluxional notation with $\dot{x}$ for dx/dt, etc., it has been shown that $\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} =$ slope of tangent at the point (t), and the slope of the normal at this point will be $-\dot{x}/\dot{y}$.

Therefore the equation of the tangent at the point (t) (i.e. the point whose parameter is t), is given by

$$y - f_2(t) = \frac{\dot{y}}{\dot{x}}\{x - f_1(t)\},$$

i.e.
$$\dot{x}\{y - f_2(t)\} = \dot{y}\{x - f_1(t)\}.$$

Similarly, the equation of the normal at the point (t) is

$$\dot{x}[x - f_1(t)] + \dot{y}[y - f_2(t)] = 0.$$

Example 2.—Find the equation of the normal at the point $t = 1$ in the case of the curve $x = t^2$, $y = 2t$ (parabola).

From the equation to the curve,

$$\dot{x} = 2t, \ \dot{y} = 2.$$

Therefore, when $t = 1$,
$$\dot{x} = 2, \quad y = 2, \quad x = 1,$$

and
$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = 1 \text{ when } t = 1.$$

Hence the slope of the normal at the point is -1, and the equation of the normal will be

$$y - 2 = -1(x - 1),$$

i.e.
$$x + y = 3.$$

4. Subtangents and Subnormals.

Let $P \equiv (x, y)$ be any point on a given curve, and PN its ordinate. PT is the tangent at P cutting OX at T and making an angle ψ with OX. PG is the normal at P cutting OX at G. Then TN is known as the *subtangent* at P, and NG is known as the *subnormal* at P.

From the diagram, $\quad TN = PN/\tan\psi.$

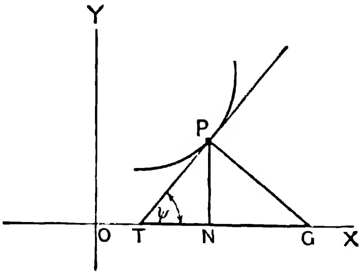
But
$$\tan\psi = \frac{dy}{dx}, \quad \therefore\ TN = y\Big/\left(\frac{dy}{dx}\right).$$

Also, by geometry, $\angle NPG = \psi$,

$$\therefore\ NG = PN \tan\psi = y\frac{dy}{dx}.$$

Hence the length of the subtangent is $y\Big/\dfrac{dy}{dx}$, and the length of the subnormal is $y\dfrac{dy}{dx}$.

Note.—If negative results be obtained, this means that T is to the right of N, and G to the left of N.



Example 3.—In the case of the parabola $y^2 = 4ax$, show that the subtangent at any point is twice the abscissa, and the subnormal is constant and equal to $2a$.

Differentiating $y^2 = 4ax$ with respect to x,

$$2y\frac{dy}{dx} = 4a, \quad \therefore y\frac{dy}{dx} = 2a. \quad \text{(i)}$$

But $y\dfrac{dy}{dx}$ is the length of the subnormal at (x, y),

$$\therefore \text{ the subnormal} = 2a = \text{constant}.$$

Also from (i), $$\frac{1}{dy/dx} = \frac{y}{2a}.$$

$$\therefore \frac{y}{dy/dx} = \frac{y^2}{2a} = \frac{4ax}{2a} \text{ (using equation of parabola)}$$
$$= 2x.$$

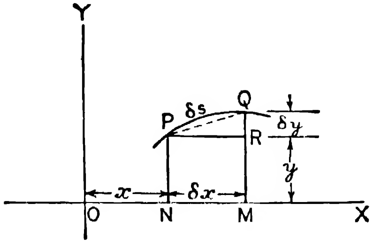
$$\therefore \text{ length of subtangent } y\Big/\frac{dy}{dx} = 2 \times \text{abscissa}.$$

5. Theorem.—If (x, y) be any point on a curve, ψ the angle the tangent at this point makes with OX, and s the length of arc to the point (x, y) from a fixed point on the curve, then:

(i) $\dfrac{ds}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$, (ii) $\dfrac{ds}{dy} = \sqrt{1 + \left(\dfrac{dx}{dy}\right)^2}$,

(iii) $\dfrac{dx}{ds} = \cos\psi$, (iv) $\dfrac{dy}{ds} = \sin\psi$.

Let $P \equiv (x, y)$, $Q \equiv (x + \delta x, y + \delta y)$ be two adjacent points on the curve. Then arc $PQ = \delta s$. PR is the perpendicular from P on the ordinate at Q.



As $\delta x \to 0$, the chord PQ $\to$ arc PQ.
Hence, using Pythagoras' theorem,

as $\delta x \to 0$, $$(\delta s)^2 \to (\delta x)^2 + (\delta y)^2, \quad \ldots \ldots \quad (1)$$

i.e. as $\delta x \to 0$, $$\left(\frac{\delta s}{\delta x}\right)^2 \to 1 + \left(\frac{\delta y}{\delta x}\right)^2,$$

i.e. $$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Also, from (1), as $\delta x \to 0$, so $\delta y \to 0$,

$$\therefore \left(\frac{\delta s}{\delta y}\right)^2 \to \left(\frac{\delta x}{\delta y}\right)^2 + 1,$$

i.e. $$\left(\frac{ds}{dy}\right)^2 = \left(\frac{dx}{dy}\right)^2 + 1.$$

$$\therefore \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}.$$

Now chord PQ $\to$ the tangent at P as $\delta x \to 0$.

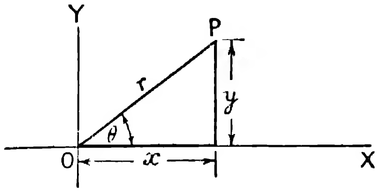
But as $\delta x \to 0$, $\qquad \dfrac{\delta x}{\delta s} \to \cos \angle QPR,$

i.e. $\qquad \dfrac{dx}{ds} = \cos \psi,$

and similarly, $\qquad \dfrac{dy}{ds} = \sin \psi.$

6. Polar Co-ordinates.

If P be any point in a plane, its position is fixed if its distance from a fixed point O in the plane is known, and also the angle that the line OP makes with some fixed line through O.



The length of OP and the angle that it makes with the fixed line through O are known as the *polar co-ordinates* of the point P. The point O is the *pole* of co-ordinates, and the fixed line through O (usually OX) is the *initial line*.

The length OP is denoted by r, and $\angle POX$ by θ. If P has polar co-ordinates (r, θ) with respect to the pole O and the initial line OX, and if P be the point (x, y) with respect to the axes of reference OX and OY, it can be seen from the diagram that $x = r\cos\theta$, $y = r\sin\theta$.

When the equation of a curve is given in the form $r = f(\theta)$ it is said to be given in the polar form, and $r = f(\theta)$ is the *polar equation* of the curve, which can be obtained from the Cartesian equation by substituting $x = r\cos\theta$, and $y = r\sin\theta$ in the Cartesian equation. The Cartesian equation can be obtained from the polar form by using $r^2 = x^2 + y^2$ and $\tan\theta = y/x$, obtained from $x = r\cos\theta$, and $y = r\sin\theta$.

Example 4.—Find the Cartesian equation corresponding to $r^2 = a^2\cos 2\theta$.

$$r^2 = a^2\cos 2\theta = a^2(\cos^2\theta - \sin^2\theta).$$

$$\therefore\ r^4 = a^2(r^2\cos^2\theta - r^2\sin^2\theta).$$

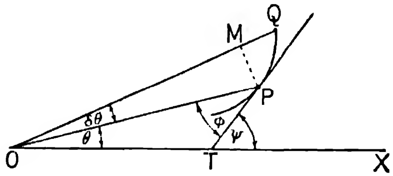
Using $r^2 = x^2 + y^2$ and $r\cos\theta = x$, $r\sin\theta = y$, this becomes

$$(x^2 + y^2)^2 = a^2(x^2 - y^2).$$

7. Theorem.—If ϕ be the angle that the tangent at $P \equiv (r, \theta)$

to the curve $r = f(\theta)$ makes with OP, to find $\tan\phi$, $\cot\phi$, $\sin\phi$, and $\cos\phi$, in terms of r and θ.

Let $Q \equiv (r + \delta r, \theta + \delta\theta)$ be an adjacent point to P on the curve. PM is the perpendicular on OQ from P, and PT is the tangent at P making an angle ψ with OX. Arc $PQ = \delta s$.



From the diagram,

$$\begin{aligned} MQ &= OQ - OM = OQ - OP\cos\delta\theta \\ &= (r + \delta r) - r\cos\delta\theta \\ &= r(1 - \cos\delta\theta) + \delta r \\ &= r\left\{\frac{(\delta\theta)^2}{2!} + \ldots\right\} + \delta r \qquad \text{(using series for } \cos\delta\theta) \\ &= \delta r \text{ to first order of small quantities.} \end{aligned}$$

Also $\quad PM = r\sin\delta\theta = r\left\{\delta\theta - \dfrac{(\delta\theta)^3}{3!} + \ldots\right\}$

$$= r\,\delta\theta \text{ to the first order of small quantities.}$$

As $\delta\theta \to 0$, the chord PQ moves into coincidence with the tangent at P, and the line OQ moves into coincidence with the line OP,

i.e. $$\underset{\delta\theta \to 0}{\mathrm{Lt}} \angle MQP = \phi.$$

Hence

$$\tan\phi = \underset{\delta\theta \to 0}{\mathrm{Lt}} \tan\angle MQP = \underset{\delta\theta \to 0}{\mathrm{Lt}} \frac{PM}{MQ} = \underset{\delta\theta \to 0}{\mathrm{Lt}}\, r\frac{\delta\theta}{\delta r} = r\frac{d\theta}{dr}.$$

$$\cot\phi = \frac{1}{\tan\phi} = \frac{1}{r}\cdot\frac{1}{d\theta/dr} = \frac{1}{r}\frac{dr}{d\theta}.$$

$$\sin\phi = \underset{\delta\theta \to 0}{\mathrm{Lt}} \sin\angle PQM = \underset{\delta\theta \to 0}{\mathrm{Lt}} \frac{PM}{PQ} = \underset{\delta s \to 0}{\mathrm{Lt}}\, r\frac{\delta\theta}{\delta s} \quad (\delta s \to 0 \text{ as } \delta\theta \to 0)$$

$$= r\frac{d\theta}{ds}.$$

$$\cos\phi = \underset{\delta\theta \to 0}{\mathrm{Lt}} \cos\angle PQM = \underset{\delta\theta \to 0}{\mathrm{Lt}} \frac{MQ}{PQ} = \underset{\delta s \to 0}{\mathrm{Lt}} \frac{\delta r}{\delta s} = \frac{dr}{ds}.$$

Using Pythagoras' theorem on ΔPMQ, in the limit as $\delta\theta \to 0$, arc PQ moves into coincidence with chord PQ.

As $\delta\theta \to 0$, $\qquad (\delta s)^2 \to (r\delta\theta)^2 + (\delta r)^2,$

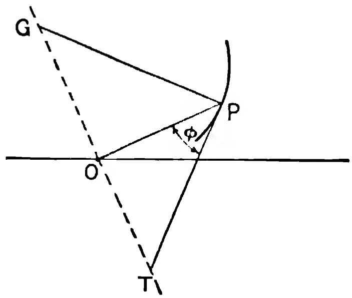
i.e. as $\delta\theta \to 0$, $\qquad \left(\dfrac{\delta s}{\delta\theta}\right)^2 \to r^2 + \left(\dfrac{\delta r}{\delta\theta}\right)^2,$

$$\therefore \quad \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2.$$

Similarly, $\qquad \left(\dfrac{ds}{dr}\right)^2 = r^2\left(\dfrac{d\theta}{dr}\right)^2 + 1.$

8. Polar subtangent, and polar subnormal.

If P be any point (r, θ) on the curve $r = f(\theta)$, with pole O and initial line OX, and the tangent at P meet the line through O per-



pendicular to OP at T, then OT is the *polar subtangent* of P. Also, if PG be the normal at P meeting TO produced at G, then OG is the *polar subnormal.*

$$\angle OPT = \phi, \quad \therefore \text{ from the previous result } \tan\phi = r\frac{d\theta}{dr}.$$

Now $\qquad \text{OT} = \text{OP}\tan\phi = r\,.\,r\dfrac{d\theta}{dr} = r^2\dfrac{d\theta}{dr}.$

$$\angle PGO = \phi \text{ by geometry}, \quad \therefore \text{OG} = \text{OP}\cot\phi = r \times \frac{1}{r}\frac{dr}{d\theta} = \frac{dr}{d\theta}.$$

Hence the length of the polar subtangent is $r^2\dfrac{d\theta}{dr}$, and that of the polar subnormal is $\dfrac{dr}{d\theta}$.

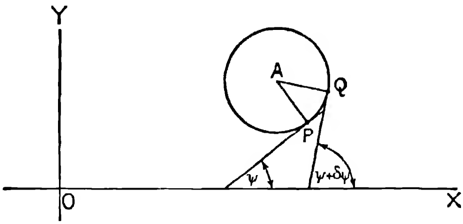
CURVATURE

9. If the tangents at two adjacent points P and Q on a curve make angles ψ and $\psi + \delta\psi$ with OX, the angle between the tangents will be $\delta\psi$.

The *curvature* at a given point of a curve is defined as being the rate of change of the angle between the tangents with respect to the arc s. Hence the curvature at P $= \underset{\delta s \to 0}{\mathrm{Lt}} \dfrac{\delta\psi}{\delta s} = \dfrac{d\psi}{ds}$.

Consider next a circle with the same adjacent points P and Q on it, the tangents at P and Q making angles ψ and $\psi + \delta\psi$ with OX, and the radius of the circle being r.

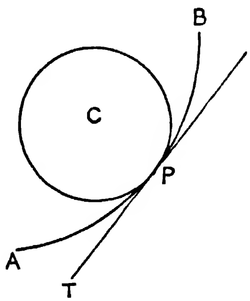
Let A be the centre of the circle.



From the properties of a circle $\angle PAQ = \delta\psi$.

But $\qquad \delta s = \text{arc PQ} = r\delta\psi, \quad \therefore \dfrac{\delta\psi}{\delta s} = \dfrac{1}{r}$.

Hence $\qquad \underset{\delta s \to 0}{\mathrm{Lt}} \dfrac{\delta\psi}{\delta s} = \dfrac{d\psi}{ds} = \dfrac{1}{r}$.



Thus the curvature at all points of a circle is the same and is equal to 1/radius.

AB is a portion of any curve and P is any point (x, y) on it. PT is the tangent at P, and C is the centre of a circle on the same side of the tangent as the curve, and having the same curvature as the curve at P. This circle is known as the *circle of curvature* of the curve at P, its centre C is the *centre of curvature* for the curve at P, and its radius CP (normal to the curve), which is usually denoted by ρ, is the *radius of curvature* of the curve at P. (The circle of curvature can

also be defined as cutting the curve in three coincident points at P.)

From the definition, and the previous result, it follows that

$$\textbf{radius of curvature } \rho = 1/\textbf{curvature} = \frac{1}{d\psi/ds} = \frac{ds}{d\psi}.$$

Radius of Curvature

10. Theorem.—To find the formula for the radius of curvature ρ at the point (x, y) of the curve given in the explicit form $y = f(x)$.

With the standard notation,

$$\tan\psi = \frac{dy}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Differentiating (2) with respect to s,

$$\sec^2\psi\frac{d\psi}{ds} = \frac{d}{ds}\left\{\frac{dy}{dx}\right\} = \left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)\right\}\frac{dx}{ds}$$

$$= \frac{d^2y}{dx^2} \times \frac{1}{\sqrt{\{1 + (dy/dx)^2\}}} \left[\text{since } \frac{dx}{ds} = 1 \Big/ \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\right].$$

$$\therefore \frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d^2y/dx^2}{\sqrt{\{1 + (dy/dx)^2\}}} \cdot \frac{1}{\sec^2\psi}$$

$$= \left[\frac{d^2y}{dx^2} \Big/ \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\right] \times \frac{1}{(1 + \tan^2\psi)}$$

$$= \left[\frac{d^2y}{dx^2} \Big/ \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\right] \times \frac{1}{1 + (dy/dx)^2}$$

$$= \frac{d^2y}{dx^2} \Big/ \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}.$$

$$\therefore \boldsymbol{\rho = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} \Big/ \frac{d^2y}{dx^2}}.$$

N.B.—The convention is that the positive root is taken in the numerator, and the radius of curvature will therefore be positive when d^2y/dx^2 is positive, i.e. when the curve is concave upwards, and negative when d^2y/dx^2 is negative, i.e. when the curve is convex upwards.

Example 5 (L.U.).—Find the point on the curve $y = e^x$ at which the curvature is a maximum, and show that the tangent at this point forms with the axes of co-ordinates a triangle whose sides are in the ratio $1 : \sqrt{2} : \sqrt{3}$.

Let C be the curvature at the point (x, y).

Now $$y = e^x, \quad \therefore \frac{dy}{dx} = e^x, \text{ and } \frac{d^2y}{dx^2} = e^x.$$

$$C = \frac{d^2y}{dx^2} \Big/ \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} = \frac{e^x}{(1 + e^{2x})^{3/2}},$$

$$\therefore \frac{dC}{dx} = \frac{e^x(1 + e^{2x})^{3/2} - \frac{3}{2}e^x(1 + e^{2x})^{1/2}\, 2 \,.\, e^{2x}}{(1 + e^{2x})^3}$$

$$= \frac{e^x + e^{3x} - 3e^{3x}}{(1 + e^{2x})^{5/2}} = \frac{e^x(1 - 2e^{2x})}{(1 + e^{2x})^{5/2}}.$$

For a turning point, $\dfrac{dC}{dx} = 0$,

i.e. $$e^x(1 - 2e^{2x}) = 0, \quad \therefore e^x = 0 \text{ or } e^{2x} = \tfrac{1}{2}.$$

$$\frac{d^2C}{dx^2} = \frac{(e^x - 6e^{3x})(1 + e^{2x})^{5/2} - (e^x - 2e^{3x})(1 + e^{2x})^{3/2} \,.\, \frac{5}{2} \,.\, 2e^{2x}}{(1 + e^{2x})^5}.$$

When $e^x = 0$, $\dfrac{d^2C}{dx^2} = 0$, $\therefore$ does not give a maximum.

When $e^{2x} = \frac{1}{2}$, $\dfrac{d^2C}{dx^2} = \dfrac{\frac{1}{\sqrt{2}}(1 - 3)(1 + \frac{1}{2})^{5/2} - 0}{(1 + \frac{1}{2})^5}$, which is negative.

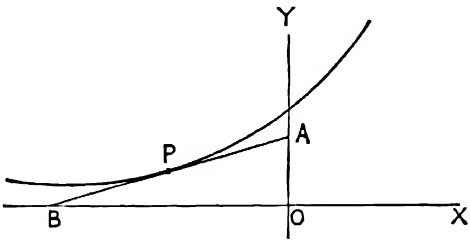
$\therefore e^{2x} = \frac{1}{2}$ gives a maximum value of C.

$\therefore C$ is a maximum when $x = \frac{1}{2} \log_e \frac{1}{2}$

$= -\frac{1}{2} \log_e 2,$

and $$y = e^x = 1/\sqrt{2},$$

i.e. at the point $(-\frac{1}{2} \log_e 2,\ 1/\sqrt{2})$.



The slope of the tangent at (x, y) is

$$\frac{dy}{dx} = e^x = y,$$

$\therefore$ slope of tangent at point P where $y = 1/\sqrt{2}$ is $1/\sqrt{2}$.

Hence equation of tangent at this point P is

$$y - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}[x + \tfrac{1}{2} \log_e 2].$$

Theorem 2.—Differential coefficient of a product.

If u and v be functions of x, y, z, etc., then

$$\frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x},$$

$$\frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}, \text{ etc.}$$

Theorem 3.—Differential coefficient of a quotient.

If u and v be functions of x, y, z, etc., then

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\dfrac{\partial u}{\partial x} - u\dfrac{\partial v}{\partial x}}{v^2},$$

$$\frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\dfrac{\partial u}{\partial y} - u\dfrac{\partial v}{\partial y}}{v^2}, \text{ etc.}$$

Theorem 4.—Function of a function.

If u be a function of p, where p is a function of x, y, z, etc., then

$$\frac{\partial}{\partial x}(u) = \frac{du}{dp}\cdot\frac{\partial p}{\partial x},$$

$$\frac{\partial}{\partial y}(u) = \frac{du}{dp}\cdot\frac{\partial p}{\partial y}, \text{ and so on.}$$

Note. $\dfrac{du}{dp}$ is used and not $\dfrac{\partial u}{\partial p}$, since u is a function of a *single* variable p.

Example 13.—Find $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial^2 u}{\partial y^2}, \dfrac{\partial^2 u}{\partial x \partial y}, \dfrac{\partial^2 u}{\partial y \partial x}$ in the following cases:

(i) $u = (x - y)/(x + y)$,

(ii) $u = x^3 \cos(y/x)$,

(iii) $u = \tan^{-1}(\log_e xy)$.

(i) $u = \dfrac{x-y}{x+y}$.

$$\therefore \frac{\partial u}{\partial x} = \frac{(x+y)\dfrac{\partial}{\partial x}(x-y) - (x-y)\dfrac{\partial}{\partial x}(x+y)}{(x+y)^2}$$

$$= \frac{(x+y)1 - (x-y)1}{(x+y)^2} = \frac{2y}{(x+y)^2}.$$

$$\frac{\partial u}{\partial y} = \frac{(x+y)\dfrac{\partial}{\partial y}(x-y) - (x-y)\dfrac{\partial}{\partial y}(x+y)}{(x+y)^2}$$

$$= \frac{(x+y)(-1) - (x-y)1}{(x+y)^2} = -\frac{2x}{(x+y)^2}.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left\{\frac{2y}{(x+y)^2}\right\} = 2y\frac{\partial}{\partial x}\{(x+y)^{-2}\}$$

$$= 2y(-2)(x+y)^{-3} = -\frac{4y}{(x+y)^3}.$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left\{-\frac{2x}{(x+y)^2}\right\} = -2x\frac{\partial}{\partial y}\{(x+y)^{-2}\}$$

$$= -2x(-2)(x+y)^{-3}$$

$$= \frac{4x}{(x+y)^3}.$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left\{\frac{2y}{(x+y)^2}\right\} = \frac{\partial}{\partial y}\{2y(x+y)^{-2}\}$$

$$= 2y\frac{\partial}{\partial y}\{(x+y)^{-2}\} + (x+y)^{-2}\frac{\partial}{\partial y}(2y)$$

$$= 2y(-2)(x+y)^{-3} + (x+y)^{-2}(2)$$

$$= -\frac{4y}{(x+y)^3} + \frac{2}{(x+y)^2} = \frac{-4y+2x+2y}{(x+y)^3} = \frac{2(x-y)}{(x+y)^3}.$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left\{-\frac{2x}{(x+y)^2}\right\} = -2\frac{\partial}{\partial x}\{x(x+y)^{-2}\}$$

$$= -2\left[x\frac{\partial}{\partial x}\{(x+y)^{-2}\} + (x+y)^{-2}\frac{\partial}{\partial x}(x)\right]$$

$$= -2\{x(-2)(x+y)^{-3} + (x+y)^{-2}(1)\}$$

$$= -2\left\{-\frac{2x}{(x+y)^3} + \frac{1}{(x+y)^2}\right\} = -2\left\{\frac{-2x+x+y}{(x+y)^3}\right\}$$

$$= \frac{2(x-y)}{(x+y)^3}.$$

(ii) $u = x^3 \cos(y/x)$.

$$\therefore \frac{\partial u}{\partial x} = x^3 \frac{\partial}{\partial x}\left(\cos\frac{y}{x}\right) + \cos\frac{y}{x}\frac{\partial}{\partial x}(x^3)$$

$$= x^3 \frac{d}{dp}(\cos p)\frac{\partial p}{\partial x} + 3x^2 \cos\frac{y}{x} \qquad \text{(where } p = y/x \text{ and } \partial p/\partial x = -y/x^2\text{)}$$

$$= x^3(-\sin y/x)(-y/x^2) + 3x^2 \cos y/x$$

$$= xy \sin y/x + 3x^2 \cos y/x = x\{y \sin y/x + 3x \cos y/x\}.$$

$$\frac{\partial u}{\partial y} = x^3 \frac{\partial}{\partial y}\left(\cos\frac{y}{x}\right) = x^3 \frac{d}{dp}(\cos p)\frac{\partial p}{\partial y} \qquad \text{(where } p = y/x \text{ and } \therefore \partial p/\partial y = 1/x\text{)}$$

$$= x^3(-\sin y/x)(1/x) = -x^2 \sin y/x.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\{xy \sin y/x + 3x^2 \cos y/x\}$$

$$= xy(\cos y/x)(-y/x^2) + (\sin y/x)y$$

$$+ 3x^2(-\sin y/x)(-y/x^2) + (3 \cos y/x)(2x)$$

$$= -y^2/x \cos y/x + y \sin y/x + 3y \sin y/x + 6x \cos y/x$$

$$= (6x - y^2/x)\cos y/x + 4y \sin y/x.$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-x^2 \sin\frac{y}{x}\right) = -x^2 \frac{\partial}{\partial y}\left(\sin\frac{y}{x}\right)$$

$$= -x^2(\cos y/x)(1/x) = -x \cos y/x.$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left\{3x^2 \cos\frac{y}{x} + xy \sin\frac{y}{x}\right\} = 3x^2 \frac{\partial}{\partial y}\left(\cos\frac{y}{x}\right) + x\frac{\partial}{\partial y}\left(y \sin\frac{y}{x}\right)$$

$$= 3x^2\{-\sin y/x\}\{1/x\} + x\{y(\cos y/x)(1/x) + \sin y/x\,(1)\}$$

$$= -3x \sin y/x + y \cos y/x + x \sin y/x$$

$$= y \cos y/x - 2x \sin y/x.$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(-x^2 \sin\frac{y}{x}\right) = (-x^2)\left(\cos\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + \sin\frac{y}{x}(-2x)$$

$$= y \cos y/x - 2x \sin y/x.$$

(iii) $u = \tan^{-1}(\log_e xy)$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{1 + (\log_e xy)^2} \cdot \frac{\partial}{\partial x}(\log_e xy) \qquad \text{(using function of a function)}$$

$$= \frac{1}{1 + (\log_e xy)^2}\left(\frac{1}{xy} \times y\right) = \frac{1}{x\{1 + (\log_e xy)^2\}}.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{1 + (\log_e xy)^2}\left(-\frac{1}{x^2}\right) - \frac{1}{[1 + (\log_e xy)^2]^2} \cdot \frac{2 \log_e xy}{x} \cdot \frac{1}{x}$$

$$= -\frac{1}{x^2[1 + (\log_e xy)^2]^2}\{1 + 2\log_e xy + (\log_e xy)^2\}.$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{1}{x}\frac{\partial}{\partial y}\left\{\frac{1}{1 + (\log_e xy)^2}\right\} = \frac{1}{x}\left\{-\frac{1}{[1 + (\log_e xy)^2]^2}\right\} \cdot \frac{2 \log_e xy}{y}$$

$$= -\frac{2 \log_e xy}{xy\{1 + (\log_e xy)^2\}^2}.$$

Since u is symmetrical in x and y, from the above results, by interchanging x and y,

$$\frac{\partial u}{\partial y} = \frac{1}{y\{1 + (\log_e xy)^2\}},$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{y^2\{1 + (\log_e xy)^2\}^2}\{1 + 2\log_e xy + (\log_e xy)^2\},$$

and
$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{2\log_e xy}{xy\{1 + (\log_e xy)^2\}^2}.$$

N.B.—In each of these examples it has been seen that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, which leads to the following theorem:

18. Theorem.—If $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y \partial x}$ are continuous in the region R, then $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ holds everywhere in this region.

From the definition of partial differentiation,

$$\frac{\partial u}{\partial x} = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}.$$

$$\therefore \frac{\partial^2 u}{\partial y \partial x} = \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{\mathrm{Lt}} \frac{\dfrac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} - \dfrac{f(x + \delta x, y) - f(x, y)}{\delta x}}{\delta y}$$

$$= \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{\mathrm{Lt}} \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y) - f(x + \delta x, y) + f(x, y)}{\delta x \,.\, \delta y}.$$

Now
$$\frac{\partial u}{\partial y} = \underset{\delta y \to 0}{\mathrm{Lt}} \frac{f(x, y + \delta y) - f(x, y)}{\delta y},$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \underset{\substack{\delta y \to 0 \\ \delta x \to 0}}{\mathrm{Lt}} \frac{\dfrac{f(x + \delta x, y + \delta y) - f(x + \delta x, y)}{\delta y} - \dfrac{f(x, y + \delta y) - f(x, y)}{\delta y}}{\delta x}$$

$$= \underset{\substack{\delta y \to 0 \\ \delta x \to 0}}{\mathrm{Lt}} \frac{f(x + \delta x, y + \delta y) - f(x + \delta x, y) - f(x, y + \delta y) + f(x, y)}{\delta x \,.\, \delta y}$$

$$= \frac{\partial^2 u}{\partial y \partial x}.$$

19. Theorem.—If $u = f(x, y, z, \ldots)$, to prove that

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy + \frac{\partial u}{\partial z} \cdot dz + \ldots,$$

where, if δu be the change in u corresponding to small changes δx, δy, δz, ... simultaneously in x, y, z, ... respectively, then du, dx, dy, etc., are the values of δx, δy, δz, ... as the limit is approached.

Note.—In the proof, only the three variables x, y, z will be used, and the result can then be extended to cover any number of variables.

$$\begin{aligned} \text{Now}\quad \delta u &= f(x + \delta x, y + \delta y, z + \delta z) - f(x, y, z) \\ &= [f(x + \delta x, y + \delta y, z + \delta z) - f(x, y + \delta y, z + \delta z)] \\ &\quad + [f(x, y + \delta y, z + \delta z) - f(x, y, z + \delta z)] \\ &\quad + [f(x, y, z + \delta z) - f(x, y, z)] \\ &= \frac{[f(x + \delta x, y + \delta y, z + \delta z) - f(x, y + \delta y, z + \delta z)]}{\delta x} \cdot \delta x \\ &\quad + \frac{[f(x, y + \delta y, z + \delta z) - f(x, y, z + \delta z)]}{\delta y} \cdot \delta y \\ &\quad + \frac{[f(x, y, z + \delta z) - f(x, y, z)]}{\delta z} \cdot \delta z. \end{aligned}$$

But as $\delta x, \delta y, \delta z \to 0$,

$$\frac{f(x + \delta x, y + \delta y, z + \delta z) - f(x, y + \delta y, z + \delta z)}{\delta x} \to \frac{f(x + \delta x, y, z) - f(x, y, z)}{\delta x} \to \frac{\partial u}{\partial x},$$

and similarly

$$\frac{f(x, y + \delta y, z + \delta z) - f(x, y, z + \delta z)}{\delta y} \to \frac{\partial u}{\partial y},$$

$$\frac{f(x, y, z + \delta z) - f(x, y, z)}{\delta z} \to \frac{\partial u}{\partial z}.$$

$$\therefore\ \boldsymbol{du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy + \frac{\partial u}{\partial z} \cdot dz.}$$

This result gives the small error du in u due to separate errors of dx, dy, dz in x, y, z respectively.

If x, y, z, . . . are all functions of a *single* variable t, by extending the above theorem to more than three variables and division by dt,

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial u}{\partial z}\cdot\frac{dz}{dt}+\cdots.$$

Note.—Ordinary differential coefficients can be used in this, since u, x, y, z, . . . are functions of a single variable t.

20. Consider now the equation $f(x, y) = 0$. From this it can be seen that y is some function of x.

Writing the equation $u = f(x, y)$, where $u = 0$, and using the previous result,

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}.$$

But $$u = 0, \qquad \therefore \frac{du}{dx} = 0,$$

i.e. $$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0$$ [f is sometimes used instead of $f(x, y)$],

i.e. $$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}.$$

This result is useful when finding the slope of the tangent at a point of a curve whose equation is given in the implicit form.

Example 14.—Find the slope of the tangent at the point (x_1, y_1) of the curve $3x^2 + 2xy + y^3 + 3 = 0$.

Let $$f(x, y) \equiv 3x^2 + 2xy + y^3 + 3.$$

$$\therefore \frac{\partial f}{\partial x} = 6x + 2y, \qquad \frac{\partial f}{\partial y} = 2x + 3y^2.$$

$$\therefore \frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} = -\frac{(6x + 2y)}{2x + 3y^2}.$$

$$\therefore \text{slope of tangent at } (x_1, y_1) \text{ is } -\frac{2(3x_1 + y_1)}{2x_1 + 3y_1^2}.$$

21. If $u = f(x, y, z, \ldots)$, where x, y, z, . . . are all functions of *several variables* w_1, w_2, w_3, . . . , dividing through the result

$$du=\frac{\partial u}{\partial x}\,.\,dx+\frac{\partial u}{\partial y}\,.\,dy+\frac{\partial u}{\partial z}\,.\,dz+\ldots$$

From (v) and (vi),

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=x^2\left\{\frac{1}{a}\frac{\partial V}{\partial u}+\frac{1}{b}\frac{\partial V}{\partial v}\right\}+y^2\left\{\frac{1}{a}\frac{\partial V}{\partial u}-\frac{1}{b}\frac{\partial V}{\partial v}\right\}$$

$$=\frac{1}{a}(x^2+y^2)\frac{\partial V}{\partial u}+\frac{1}{b}(x^2-y^2)\frac{\partial V}{\partial v}$$

$$=2u\frac{\partial V}{\partial u}+2v\frac{\partial V}{\partial v} \qquad \text{[using (iii) and (iv)]}$$

$$=2\left\{u\frac{\partial V}{\partial u}+v\frac{\partial V}{\partial v}\right\}.$$

EXAMPLES ON CHAPTER VII

All the following questions are taken from London University papers.

1. If OY_1, OY_2 are the perpendiculars from O on the tangent and normal respectively at the point $(a\cos^3\theta,\ a\sin^3\theta)$ on the curve $x^{2/3}+y^{2/3}=a^{2/3}$, prove that $4OY_1^2+OY_2^2=a^2$.

Show that the equation of the locus of the point of intersection of perpendicular tangents to the curve is $2(x^2+y^2)^3=a^2(x^2-y^2)^2$.

2. Show that the curve $y^2=a^2(x-a)/x$ has the lines $y=\pm a$ and $x=0$ as asymptotes. Sketch the curve.

Find the equations of the tangents which pass through the origin, and deduce, or prove otherwise, that the equation in x,

$$\lambda x^3=a^2(x-a),$$

has three real roots if $0<\lambda<4/27$, but only one real root if λ does not lie within these limits.

3. If ϕ be one of the angles between the tangent at a point P of a plane curve and the radius vector drawn from the origin to P, prove that $\cos\phi=dr/ds$ and $\sin\phi=r\,d\theta/ds$, making it clear which of the two angles is ϕ.

O, the origin, is the cusp, and A, the point $(2a, 0)$, is the vertex of the cardioid whose equation is $r=a(1+\cos\theta)$. P is any point on the cardioid, and OP is produced beyond P to meet at Q the circle with O as centre and OA as radius. Prove that (i) the chord AQ and the tangent to the cardioid at P are equally inclined to OP, (ii) the length of the chord AQ is equal to the length of the arc AP of the cardioid.

4. Find the equation to the tangent at the point (a, a^3) on the curve $y=x^3$. Show that the tangent meets the curve again at the point $(-2a, -8a^3)$, and that the area included between the tangent and the curve is $27a^4/4$.

5. A circle of radius a rolls externally on a fixed circle of radius $2a$. Show that, when referred to axes through the centre of the fixed circle, the equations to the curve described by a point on the circumference of the rolling circle can be written in the form

$$x=3a\cos\theta-a\cos 3\theta,\quad y=3a\sin\theta-a\sin 3\theta.$$

Prove also that in this curve $p = 4a \sin \psi/2$, where p is the perpendicular from the origin on the tangent and ψ is the angle that the tangent makes with the axis OX of the above co-ordinates.

6. The parametric co-ordinates of a point on a curve are given by

$$2x = 3a \cos \theta + a \cos 3\theta, \quad 2y = 3a \sin \theta + a \sin 3\theta.$$

Find the length of the perpendicular from the origin to the tangent to this curve at the point (θ). Show that, with the origin as pole, the (p, r) equation of the curve is $4r^2 = 3p^2 + 4a^2$.

7. If the co-ordinates of a point on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ are represented parametrically in the form $x = a \sin^3 t$, $y = a \cos^3 t$, show that t is the angle which the perpendicular drawn from O to the tangent makes with OX.

If p be the length of this perpendicular, show that the length of the radius of curvature is $3p$.

8. Show that the locus of the foot of the perpendicular from the pole upon a tangent to the curve $r = a(1 + \cos \theta)$ is the curve $r = 2a \cos^3 \theta/3$, when referred to the same pole and initial line.

Show that the radius of curvature at any point of this locus is $\frac{3}{4}$ of the length of the radius vector to the corresponding point on the original curve.

9. Show that $$\rho = \pm \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} \Big/ \frac{d^2y}{dx^2}$$

for a plane curve, the upper or lower sign being taken according as the curve is convex or concave to the axis of x.

The normal at the point P of the conic $2abx = bx^2 + ay^2$ meets the axis of x at G. Show that the length of the radius of curvature at P is $\mathrm{PG}^3 \div b^2$.

10. Prove, with the usual notation, that the radius of curvature ρ at a point on a plane curve is $r\,dr/dp$.

Draw a rough sketch of the curve $r^3 = a^3 \sin 3\theta$, showing that it consists of three loops. If P be the point (r, θ) on the curve, show that the tangent at P makes an angle 4θ with the initial line, and that $\rho = a^3/4r^2$.

If the circle of curvature for the point P intersects OP at Q, where O is the pole, find the ratio of OQ to OP.

11. With the usual notation, show that the radius of curvature is given by

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} \Big/ \frac{d^2y}{dx^2}.$$

Find the radius of curvature and the co-ordinates of the centre of curvature at the point P of the curve $y = a \cosh x/a$. If the normal at P meets the x-axis at G, show that GP is equal to the radius of curvature.

12. With the usual notation show that the radius of curvature of a curve is given by $\rho = r\,dr/dp$.

P is any point on the curve whose equation is $r = ae^{\theta \cot \alpha}$, and I is its centre of curvature. Find the length of PI and show that it subtends a right angle at the origin. Find the locus of I, and the radius of curvature of this locus at I, in terms of OI.

13. Prove that the radius of curvature at the point θ on the curve whose equations are

$$x = a \sin 2\theta(1 + \cos 2\theta), \quad y = a \cos 2\theta(1 - \cos 2\theta)$$

is $4a \cos 3\theta$.

14. A circle of radius a rolls on the outside of a fixed circle of radius $3a$, whose centre is A, without slipping. Show by a diagram the direction of the tangent and normal to the locus of a point P on the rolling circle.

If p be the length of the perpendicular from A to the tangent at P, and r be the length of AP, show that $r^2 = 9a^2 + 16p^2/25$, and that the radius of curvature at P is $16p/25$.

15. Prove the formula $\rho = (\dot{x}^2 + \dot{y}^2)^{3/2}/(\dot{x}\ddot{y} - \ddot{x}\dot{y})$ for the radius of curvature of a curve whose co-ordinates are given as functions of a parameter t, differentiations with respect to t being denoted by dots.

Find the radius and centre of curvature at any point of the cycloid

$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$

16. Prove, from first principles, that the radius of curvature at the point θ on the curve

$$x = 3a \cos \theta - a \cos 3\theta, \quad y = 3a \sin \theta - a \sin 3\theta$$

is $3a \sin \theta$.

17. (a) If $x = r \cos \theta$, and $y = r \sin \theta$, where (x, y) and (r, θ) are pairs of independent variables, show that

$$/\partial r\backslash \qquad /\partial x\backslash$$
$$\partial x/\,y$$

(b) If $x = u + v$, $y = uv$, and Z be any function of x and y, show that

$$u\frac{\partial Z}{\partial u} + v\frac{\partial Z}{\partial v} = x\frac{\partial Z}{\partial x} + 2y\frac{\partial Z}{\partial y}.$$

18. (i) Prove that, if $u = x^n f(Y)$, where $Y = y/x$, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

If $u = x^2 \tan^{-1} y/x - y^2 \tan^{-1} x/y$, evaluate $x\,\partial u/\partial x + y\,\partial u/\partial y$.

(ii) If $u = x^3 - 3xy^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Prove also that, if $z = r^n u$, where $r^2 = x^2 + y^2$, then

$$^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = n^2 z + 6nz.$$

19. The sides b, c and the angle A of a triangle ABC are measured, and the angles B and C are calculated accurately from these measurements. Show by means of the formula

$$\tan \tfrac{1}{2}(B - C) = \frac{b - c}{b + c} \cot \tfrac{1}{2}A,$$

or otherwise, that small errors δB and δC in B and C, due to small errors ∂b, δc, δA in the measurements of b, c, and A, are given by

$$\delta B = (c\,\delta b - b\,\delta c)\sin A/a^2 - b\cos C\,.\,\delta A/a,$$
$$\delta C = (b\,\delta c - c\,\delta b)\sin A/a^2 - c\cos B\,.\,\delta A/a.$$

20. (i) Find dy/dx and d^2y/dx^2 when $x^4 + y^4 = 5a^2xy$.

(ii) Change the independent variables x and y in the equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$$

to u and v, where $u = x + y$, and $v = x - y$.

21. The variables P, V, T, ϕ are connected by the relations

$$PV = RT, \quad \phi = C_v \log P + C_p \log V,$$

where C_p, C_v, and R are constants, and $C_p - C_v = R$. Prove that

$$\frac{\partial T}{\partial P}\cdot\frac{\partial \phi}{\partial V} - \frac{\partial T}{\partial V}\cdot\frac{\partial \phi}{\partial P} = \frac{\partial P}{\partial T}\cdot\frac{\partial V}{\partial \phi} - \frac{\partial P}{\partial \phi}\cdot\frac{\partial V}{\partial T} = 1.$$

22. (*a*) If $V = f(x^2 + y^2)$, where f is any function, show that

(i) $y\dfrac{\partial V}{\partial x} - x\dfrac{\partial V}{\partial y} = 0,$

(ii) $y^2\dfrac{\partial^2 V}{\partial x^2} - 2xy\dfrac{\partial^2 V}{\partial x\partial y} + x^2\dfrac{\partial^2 V}{\partial y^2} = x\dfrac{\partial V}{\partial x} + y\dfrac{\partial V}{\partial y}.$

(*b*) Show that, if $u = \log_e r$, where $r^2 = x^2 + y^2$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

23. If $u = f(x, y)$, show, with the usual notation, that

$$du = \frac{\partial u}{\partial x}\,.\,dx + \frac{\partial u}{\partial y}\,.\,dy.$$

In a triangle ABC, the angle A is accurately known, but the measurement of the side b may be in error to the extent δb, and that of the side c to the extent δc. Find the error obtained on calculating the value of a from b, c, and A.

What is the best shape of the triangle ABC in order to minimize as much as possible the effect of the error δb.

24. If $u = (x^2 - y^2)f(t)$, where $t = xy$, prove that

$$\frac{\partial^2 u}{\partial x\partial y} = (x^2 - y^2)\{tf''(t) + 3f'(t)\}.$$

(*b*) Prove that $\qquad V = \dfrac{1}{r}\{\phi(ct + r) + \psi(ct - r)\}$

satisfies the equation $\qquad \dfrac{\partial^2 V}{\partial t^2} = \dfrac{c^2}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial V}{\partial r}\right).$

25. If $u = x^n f(y/x)$, where f denotes an arbitrary function, show that

$$\text{(i)} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu,$$

$$\text{(ii)} \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

26. If $V = (x^2 - y^2)\,f(xy)$, show that

$$\text{(i)} \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = (x^4 - y^4)f''(xy),$$

$$\text{(ii)} \quad \frac{\partial^2 V}{\partial x \partial y} = (x^2 - y^2)\{3f'(xy) + xyf''(xy)\}.$$

27. If $V = f(z)$, where $z^2 = t^2 - x^2$, and x and t are independent variables,

$$z\frac{\partial V}{\partial t} = t\frac{df}{dz}.$$

Show that, if V satisfies the equation

$$\frac{\partial^2 V}{\partial x^2} - V = \frac{\partial^2 V}{\partial t^2},$$

then $f(z)$ is a solution of the equation

$$\frac{d^2 f}{dz^2} + \frac{1}{z}\frac{df}{dz} + f = 0.$$

28. (i) If the four variables x, y, z, u are connected by the two relations

$$u = \sin x \cosh y, \quad \log_e(x + y) + 2y - 3\log_e z = 4,$$

find $\partial u/\partial x$ when x and z are the independent variables.

(ii) If $x = r\cos\theta$, $y = r\sin\theta$, transform the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

to the independent variables r and θ.

29. Find the value of a, if $V = x^3 + axy^2$ satisfies the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

Taking this value of V, show that if $U = r^n V$, and $r^2 = x^2 + y^2$, then

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = n(n + 6)r^{n-2}V.$$

30. (i) Find the value of μ in order that $V = t^{\mu}e^{-r^2/4t}$ shall satisfy the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = \frac{\partial V}{\partial t}.$$

(ii) If V be a given function of x and y, and u and v are defined in terms of x and y by the equations $u = e^{xv}$, $v = x^2 + y^2$, express $\partial V/\partial x$, $\partial V/\partial y$ in terms of $\partial V/\partial u$, $\partial V/\partial v$, x and y.

CHAPTER VIII

Determinants

1. Consider the equations

$$a_2x + b_2y + c_2z = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$a_3x + b_3y + c_3z = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Solving these,

$(1) \times c_3$ $$a_2c_3x + b_2c_3y + c_2c_3z = 0. \quad . \quad . \quad . \quad (3)$$

$(2) \times c_2$ $$a_3c_2x + b_3c_2y + c_3c_2z = 0. \quad . \quad . \quad . \quad (4)$$

$(3) - (4)$ gives $$x(a_2c_3 - a_3c_2) + y(b_2c_3 - b_3c_2) = 0,$$

i.e. $$\frac{x}{b_2c_3 - b_3c_2} = -\frac{y}{a_2c_3 - a_3c_2}.$$

Similarly, from $(1) \times b_3 - (2) \times b_2$,

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{z}{a_2b_3 - a_3b_2},$$

$$\therefore \frac{x}{b_2c_3 - b_3c_2} = -\frac{y}{a_2c_3 - a_3c_2} = \frac{z}{a_2b_3 - a_3b_2}. \quad . \quad . \quad (5)$$

The quantities $(b_2c_3 - b_3c_2)$, $(a_2c_3 - a_3c_2)$, $(a_2b_3 - a_3b_2)$ are usually written in the form

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \text{ respectively,}$$

and these are known as *second order determinants*.

If, in addition, the variables x, y, z also satisfy the equation

$$a_1x + b_1y + c_1z = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

using (5) in (6), the necessary condition is

$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = 0. \quad (7)$$

This condition is known as the *eliminant* of the three equations (1), (2), and (6).

The equation (7) can also be written in the determinant form

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0, \quad . \quad . \quad (8)$$

or in the expanded form

$$a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 = 0. \quad (9)$$

The left-hand side of any of the equations (7), (8), or (9) is written more briefly in the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

which is known as a *determinant of the third order* and is usually denoted by the symbol Δ.

N.B.—A determinant has the same number of rows as columns.

It can be seen from the previous discussion, that the expansion of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is obtained by multiplying a_1 by the determinant obtained by omitting the row and column containing a_1, subtracting from this the product of b_1 and the determinant obtained by omitting the row and column containing b_1, and adding to the result the product of c_1 and the determinant obtained by omitting the row and column containing c_1.

The quantities a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 are known as the *elements* of the determinant. The determinant obtained by omitting the row and column containing a certain element is known as the *minor* of that element. Thus, in the third order determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ the determinant} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

is known as the minor of a_1 and is denoted by the symbol A_1;

$\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$ is the minor of b_2 and is denoted by B_2;

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is the minor of c_3 and is denoted by C_3; and so on.

N.B.—From the equation (9) it can be seen that each term in the expansion of the determinant contains one element, and only one, from each row and column of the determinant.

A fourth order determinant is a determinant containing four rows and four columns and, in general, an nth order determinant contains n rows and n columns and, in the expansion of an nth order determinant whose first row is $a_1, a_2, a_3, \ldots, a_n$, the minors of these being $A_1, A_2, A_3, \ldots, A_n$ respectively (with the same meaning for *minor* as in the third order determinant), the value of the determinant is

$$a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4 + \ldots + (-1)^{n+1}a_nA_n.$$

Usually nothing beyond the fourth order determinant will be required.

Example 1.—Expand (i) $\Delta = \begin{vmatrix} 24 & 17 \\ 11 & 25 \end{vmatrix}$, (ii) $\Delta = \begin{vmatrix} 2x - 3y & x + 2y \\ 4x + 5y & 3x + 5y \end{vmatrix}$.

$$\text{(i) } \Delta = 24 \times 25 - 17 \times 11 = 600 - 187 = 413.$$

$$\begin{aligned} \text{(ii) } \Delta &= (2x - 3y)(3x + 5y) - (x + 2y)(4x + 5y) \\ &= 2x^2 - 12xy - 25y^2. \end{aligned}$$

Example 2.—Solve the equation $\begin{vmatrix} 2x & 5 \\ 9 & x + 3 \end{vmatrix} = \begin{vmatrix} 5 & 4 \\ 13 & 3x \end{vmatrix}$.

On expansion the equation becomes

$$2x(x + 3) - 5 \times 9 = 5 \times 3x - 4 \times 13,$$

i.e. $$2x^2 + 6x - 45 = 15x - 52,$$

i.e. $$2x^2 - 9x + 7 = 0.$$

$$\therefore (2x - 7)(x - 1) = 0,$$

$$\therefore x = 1 \text{ or } 7/2.$$

Example 3.—Evaluate (i) $\Delta = \begin{vmatrix} 3 & -4 & -3 \\ 2 & 7 & -31 \\ 5 & -9 & 2 \end{vmatrix}$,

(ii) $\Delta = \begin{vmatrix} 2 & 3 & 5 & -1 \\ 1 & 2 & -3 & 2 \\ 4 & -1 & 2 & 5 \\ 2 & -2 & 3 & 1 \end{vmatrix}$, (iii) $\Delta = \begin{vmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \\ 4 & 6 & 8 & 10 \end{vmatrix}$.

$$\begin{aligned} \text{(i) } \Delta &= 3\begin{vmatrix} 7 & -31 \\ -9 & 2 \end{vmatrix} - (-4)\begin{vmatrix} 2 & -31 \\ 5 & 2 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 7 \\ 5 & -9 \end{vmatrix} \\ &= 3(14 - 279) + 4(4 + 155) - 3(-18 - 35) \\ &= 3(-265) + 4(159) - 3(-53) = -795 + 636 + 159 \\ &= 0. \end{aligned}$$

$$\text{(ii)}\ \Delta = 2\begin{vmatrix} 2 & -3 & 2 \\ -1 & 2 & 5 \\ -2 & 3 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & -3 & 2 \\ 4 & 2 & 5 \\ 2 & 3 & 1 \end{vmatrix}$$

$$+ 5\begin{vmatrix} 1 & 2 & 2 \\ 4 & -1 & 5 \\ 2 & -2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 2 & -3 \\ 4 & -1 & 2 \\ 2 & -2 & 3 \end{vmatrix}$$

$$= 2\left\{2 \begin{matrix} 2 & 5 \\ 3 & 1 \end{matrix} + 3 \begin{matrix} -1 & 5 \\ -2 & 1 \end{matrix} + 2 \begin{matrix} -1 \\ -2 \end{matrix} \right\}$$

$$- 3\left\{1 \begin{matrix} 2 & 5 \\ 3 & 1 \end{matrix} + 3 \begin{matrix} 4 & 5 \\ 2 & 1 \end{matrix} + 2 \begin{matrix} 4 & 2 \\ 2 & 3 \end{matrix} \right.$$

$$+ 5\left\{1 \begin{matrix} -1 \\ -2 \end{matrix} \qquad \begin{matrix} 4 & 5 \\ 2 & 1 \end{matrix} + 2 \begin{matrix} 4 & -1 \\ 2 & -2 \end{matrix} \right.$$

$$+ 1\left\{1 \begin{vmatrix} -1 & 2 \\ -2 & 3 \end{matrix} - 2 \begin{matrix} 4 & 2 \\ 2 & 3 \end{matrix} - 3 \begin{matrix} 4 & -1 \\ 2 & -2 \end{matrix} \right.$$

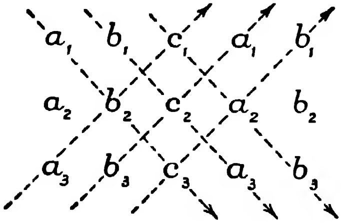
$$\begin{aligned} &= 2\{2(2 - 15) + 3(-1 + 10) + 2(-3 + 4)\} \\ &\quad - 3\{1(2 - 15) + 3(4 - 10) + 2(12 - 4)\} \\ &\quad + 5\{1(-1 + 10) - 2(4 - 10) + 2(-8 + 2)\} \\ &\quad + 1\{1(-3 + 4) - 2(12 - 4) - 3(-8 + 2)\} \\ &= 2\{-26 + 27 + 2\} - 3\{-13 - 18 + 16\} \\ &\quad + 5\{9 + 12 - 12\} + 1\{1 - 16 + 18\} \\ &= 6 + 45 + 45 + 3 \\ &= 99. \end{aligned}$$

$$\text{(iii)}\ \Delta = 1\begin{vmatrix} 4 & 6 & 8 \\ 5 & 7 & 9 \\ 6 & 8 & 10 \end{vmatrix} - 3\begin{vmatrix} 2 & 6 & 8 \\ 3 & 7 & 9 \\ 4 & 8 & 10 \end{vmatrix} + 5\begin{vmatrix} 2 & 4 & 8 \\ 3 & 5 & 9 \\ 4 & 6 & 10 \end{vmatrix} - 7\begin{vmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \\ 4 & 6 & 8 \end{vmatrix}$$

$$\begin{aligned} &= 1\{4(70 - 72) - 6(50 - 54) + 8(40 - 42\} \\ &\quad - 3\{2(70 - 72) - 6(30 - 36) + 8(24 - 28)\} \\ &\quad + 5\{2(50 - 54) - 4(30 - 36) + 8(18 - 20)\} \\ &\quad - 7\{2(40 - 42) - 4(24 - 28) + 6(18 - 20)\} \\ &= \{-8 + 24 - 16\} - 3\{-4 + 36 - 32\} \\ &\quad + 5\{-8 + 24 - 16\} - 7\{-4 + 16 - 12\} \\ &= 0. \end{aligned}$$

2. The rule of Sarrus.

This is a rule which can be used for writing down the expansion of third order determinants.



The above diagram exemplifies the rule in the case of the expansion of the determinant

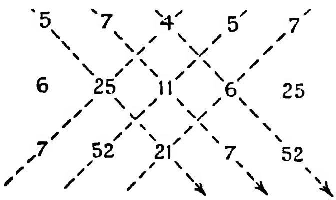
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and the procedure is as follows:

Write down the three columns of the determinant and repeat the first two. Draw lines diagonally through each of three elements as shown. Then the terms formed by each line will be terms of the expansion, and those with arrows downwards will be positive, and those with arrows upwards will be negative.

Example 4.—Evaluate $\Delta = \begin{vmatrix} 5 & 7 & 4 \\ 6 & 25 & 11 \\ 7 & 52 & 21 \end{vmatrix}$ by the rule of Sarrus.

Using the rule of Sarrus,



$$\begin{aligned} \Delta &= 5.25.21 + 7.11.7 + 4.6.52 - 4.25.7 - 5.11.52 - 7.6.21 \\ &= 2625 + 539 + 1248 - 700 - 2860 - 882 \\ &= -30. \end{aligned}$$

Most determinants can be simplified before evaluation, and the following theorems can be used for this simplification.

These theorems are only proved for third order determinants, but can be extended to cover all determinants.

3. Theorem.—A determinant is unaltered in value if the rows are changed to columns.

Consider
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then, with the usual notation for minors,
$$\Delta = a_1A_1 - b_1B_1 + c_1C_1.$$

Now
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$
$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$
(rearranging terms)
$$= a_1A_1 - b_1B_1 + c_1C_1 = \Delta,$$

and thus the theorem is proved.

From this result,
$$\Delta = a_1A_1 - b_1B_1 + c_1C_1$$
$$= a_1A_1 - a_2A_2 + a_3A_3,$$

and therefore the determinant can be expanded by columns instead of rows.

4. Theorem.—If two rows (or columns) of a determinant be interchanged, the determinant changes sign.

Let
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 - b_1B_1 + c_1C_1.$$

Interchanging the first and second rows,
$$\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
$$= a_2(b_1c_3 - b_3c_1) - b_2(a_1c_3 - a_3c_1) + c_2(a_1b_3 - a_3b_1)$$
$$= -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$$
(rearranging terms)
$$= -\{a_1A_1 - b_1B_1 + c_1C_1\} = -\Delta.$$

A similar result can be obtained by interchanging two columns, and thus the theorem is proved.

N.B.—In the case of two interchanges the determinant will change its sign twice and hence will remain at its original value. If three interchanges be made the result will be the original determinant with a negative sign.

5. Theorem.—If two rows (or columns) of a determinant are the same the value of the determinant is zero.

Consider the determinant $\Delta = \begin{array}{|ccc} & b_1 & \\ & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array}$ having two rows equal.

Interchanging the equal rows we have

$$\begin{array}{ccc|} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array} = \Delta. \qquad (10)$$

But, by the previous theorem, the L.H.S. of (10) is $-\Delta$,

$$\therefore\ \Delta = -\Delta, \qquad \therefore\ \Delta = 0.$$

The same result will be obtained when two columns are equal.

6. Theorem.—If the elements of any row (or column) of a determinant be each multiplied by the same factor, the result is the product of that factor and the original determinant,

i.e. $$\begin{vmatrix} pa_1 & pb_1 & pc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = p \begin{array}{|ccc} a_1 & b_1 & c_1 \\ & & \\ & & \end{array}$$

With the usual notation, where

$$\Delta = \begin{array}{ccc} & b_1 & \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}$$

$$\begin{array}{ccc|} pa_1 & pb_1 & pc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} = pa_1(b_2c_3 - b_3c_2) - pb_1(a_2c_3 - a_3c_2) + pc_1(a_2b_3 - a_3b)_2$$

$$= p\{a_1A_1 - b_1B_1 + c_1C_1\} = p\Delta.$$

The same result can be proved for the case of the columns.

7. Theorem.—If each element of any row (or column) of a determinant consists of the algebraic sum of r terms, the determinant is equivalent to the sum of r other determinants in each of which the elements consist of single terms.

Let A_1, B_1, C_1 be the minors of a_1, b_1, c_1 respectively in the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Then, expanding by the first column, the determinant

$$\begin{vmatrix} a_1 + l_1 - m_1 & a_2 & a_3 \\ b_1 + p_1 - q_1 & b_2 & b_3 \\ c_1 + r_1 - s_1 & c_2 & c_3 \end{vmatrix}$$

$$= (a_1 + l_1 - m_1)A_1 - (b_1 + p_1 - q_1)B_1 + (c_1 + r_1 - s_1)C_1$$

$$= (a_1A_1 - b_1B_1 + c_1C_1) + (l_1A_1 - p_1B_1 + r_1C_1) - (m_1A_1 - q_1B_1 + s_1C_1)$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} l_1 & a_2 & a_3 \\ p_1 & b_2 & b_3 \\ r_1 & c_2 & c_3 \end{vmatrix} - \begin{vmatrix} & a_2 & a_3 \\ & b_2 & b_3 \\ & c_2 & c_3 \end{vmatrix}$$

which proves the theorem for $r = 3$.

This can be extended to cover the case when the elements of two different columns consist of two or more terms as follows:

Let A_1, A_2., etc., be the minors of a_1, a_2, etc., respectively, in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

en

$$\begin{vmatrix} a_1 + \alpha_1 & b_1 + \beta_1 & c_1 \\ a_2 + \alpha_2 & b_2 + \beta_2 & c_2 \\ a_3 + \alpha_3 & b_3 + \beta_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 + \beta_1 & c_1 \\ a_2 & b_2 + \beta_2 & c_2 \\ a_3 & b_3 + \beta_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 + \beta_1 & c_1 \\ \alpha_2 & b_2 + \beta_2 & c_2 \\ \alpha_3 & b_3 + \beta_3 & c_3 \end{vmatrix}$$

(by previous result)

$$- \begin{vmatrix} b_1 + \beta_1 & a_1 & c_1 \\ b_2 + \beta_2 & a_2 & c_2 \\ b_3 + \beta_3 & a_3 & c_3 \end{vmatrix} - \begin{vmatrix} b_1 + \beta_1 & \alpha_1 & c_1 \\ b_2 + \beta_2 & \alpha_2 & c_2 \\ b_3 + \beta_3 & \alpha_3 & c_3 \end{vmatrix}$$

(interchanging first two columns)

$$\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} - \begin{vmatrix} \beta_1 & a_1 & c_1 \\ \beta_2 & a_2 & c_2 \\ \beta_3 & a_3 & c_3 \end{vmatrix} - \begin{vmatrix} b_1 & \alpha_1 & c_1 \\ b_2 & \alpha_2 & c_2 \\ b_3 & \alpha_3 & c_3 \end{vmatrix} - \begin{vmatrix} \beta_1 & \alpha_1 & c_1 \\ \beta_2 & \alpha_2 & c_2 \\ \beta_3 & \alpha_3 & c_3 \end{vmatrix}$$

(using previous result)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & \beta_1 & c_1 \\ a_2 & \beta_2 & c_2 \\ a_3 & \beta_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & & c_2 \\ & b_3 & \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & c_1 \\ \alpha_2 & \beta_2 & c_2 \\ \alpha_3 & \beta_3 & c_3 \end{vmatrix}$$

(interchanging two columns in each determinant).

This result can be further extended to the case when there are two or more terms in some or all of the elements of the determinant.

8. Theorem.—If the elements of any row (or column) be increased or diminished by equimultiples of the corresponding elements of any other row (or column) the value of the determinant is unaltered.

In algebraical language, this means that

$$\begin{vmatrix} a_1 + ub_1 + vc_1 & b_1 & c_1 \\ a_2 + ub_2 + vc_2 & b_2 & c_2 \\ a_3 + ub_3 + vc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

where u and v can take any values.

Using the previous theorem,

$$\begin{vmatrix} a_1 + ub_1 + vc_1 & b_1 & c_1 \\ a_2 + ub_2 + vc_2 & b_2 & c_2 \\ a_3 + ub_3 + vc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ub_1 & b_1 & c_1 \\ ub_2 & b_2 & c_2 \\ ub_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} vc_1 & b_1 & c_1 \\ vc_2 & b_2 & c_2 \\ vc_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} & b_1 & c_1 \\ & b_2 & c_2 \\ & b_3 & c_3 \end{vmatrix} + u \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} + v \begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}.$$

The last two determinants have each two columns identical and therefore have zero value.

Hence the original determinant $= \begin{vmatrix} a_1 & b_1 & c_1 \\ & & c_2 \\ a_3 & & c_3 \end{vmatrix}$

A similar result can be obtained by using rows instead of columns.

Note.—This theorem is very useful for simplifying determinants before evaluation, and quicker simplification can be obtained by using the fact that one column can be combined with each of two other columns simultaneously.

Thus
$$\begin{vmatrix} a_1 + rc_1 & b_1 + sc_1 & c_1 \\ a_2 + rc_2 & b_2 + sc_2 & c_2 \\ a_3 + rc_3 & b_3 + sc_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} rc_1 & b_1 & c_1 \\ rc_2 & b_2 & c_2 \\ rc_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} rc_1 & sc_1 & c_1 \\ rc_2 & sc_2 & c_2 \\ rc_3 & sc_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & sc_1 & c_1 \\ a_2 & sc_2 & c_2 \\ a_3 & sc_3 & c_3 \end{vmatrix}$$
(Section 7)

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + r\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} + rs\begin{vmatrix} c_1 & c_1 & c_1 \\ c_2 & c_2 & c_2 \\ c_3 & c_3 & c_3 \end{vmatrix} + s\begin{vmatrix} a_1 & c_1 & c_1 \\ a_2 & c_2 & c_2 \\ a_3 & c_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ (since the last 3 determinants are zero, having 2 columns identical).

It must be carefully noted that one column, when using these theorems, must always be kept in its original form or an impossible result will be obtained.

For, consider
$$\begin{vmatrix} a_1 + rb_1 & b_1 + sc_1 & c_1 + ma_1 \\ a_2 + rb_2 & b_2 + sc_2 & c_2 + ma_2 \\ a_3 + rb_3 & b_3 + sc_3 & c_3 + ma_3 \end{vmatrix}.$$

The expansion of this will involve the sum of six determinants, four of which are zero, having two columns identical, and the other two being

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} rb_1 & sc_1 & ma_1 \\ rb_2 & sc_2 & ma_2 \\ rb_3 & sc_3 & ma_3 \end{vmatrix}.$$

The latter of these two determinants is equal to

$$rms\begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix} = rms\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

which is obviously not zero unless r, s, or m, or the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

has a zero value. Thus the original determinant in this case is not generally equal to the determinant Δ.

9. Solution of a system of simultaneous linear equations by determinants.

Let the given equations be

$$a_1x + b_1y + c_1z + k_1 = 0. \quad . \quad . \quad . \quad . \quad (11)$$
$$a_2x + b_2y + c_2z + k_2 = 0. \quad . \quad . \quad . \quad . \quad (12)$$
$$a_3x + b_3y + c_3z + k_3 = 0. \quad . \quad . \quad . \quad . \quad (13)$$

A_1, A_2, A_3, etc., are the minors of a_1, a_2, a_3, etc., respectively, in the determinant

$$\Delta_0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Taking $(11) \times A_1 - (12) \times A_2 + (13) \times A_3$,

$$x(a_1A_1 - a_2A_2 + a_3A_3) + y(b_1A_1 - b_2A_2 + b_3A_3)$$
$$+ z(c_1A_1 - c_2A_2 + c_3A_3) + (k_1A_1 - k_2A_2 + k_3A_3) = 0,$$

$$\text{i.e. } x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + y\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} + z\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix} = 0.$$

The middle two determinants are zero, having two columns the same,

$$\therefore \Delta_0x + \Delta_1 = 0, \text{ where } \Delta_1 = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 & k_1 \\ b_2 & c_2 & k_2 \\ b_3 & c_3 & k_3 \end{vmatrix},$$

$$\therefore \frac{x}{\Delta_1} = -\frac{1}{\Delta_0}.$$

Similarly, by taking $(11) \times B_1 - (12) \times B_2 + (13) \times B_3$, it can be shown that

$$\frac{y}{\Delta_2} = +\frac{1}{\Delta_0}, \text{ where } \Delta_2 = \begin{vmatrix} a_1 & c_1 & k_1 \\ a_2 & c_2 & k_2 \\ a_3 & c_3 & k_3 \end{vmatrix}. = -\begin{vmatrix} c_1 & k_1 \\ c_2 & k_2 \\ a_3 & k_3 \end{vmatrix}$$

Also, by taking $(11) \times C_1 - (12) \times C_2 + (13) \times C_3$, it is found that

$$\frac{z}{\Delta_3} = -\frac{1}{\Delta_0}, \text{ where } \Delta_3 = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}. =$$

Hence the solution is $\dfrac{x}{\Delta_1} = -\dfrac{y}{\Delta_2} = \dfrac{z}{\Delta_3} = -\dfrac{1}{\Delta_0}$.

This result can be extended to n equations in n unknowns, where n is any integer.

10. Elimination.

When dealing with the equations

$$a_1x + b_1y + c_1z = 0,$$
$$a_2x + b_2y + c_2z = 0,$$
$$a_3x + b_3y + c_3z = 0,$$

at the beginning of this chapter, it was found that, in order to obtain unique values of x, y and z to satisfy all the three equations, a certain relationship between the coefficients in the equations was necessary, namely,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

This is known as the *eliminant* of the given equations. (*N.B.*—There are really only two independent variables in the equations, namely, x/z and y/z.)

A system of linear equations in n independent unknowns will in general give unique values for the unknowns, provided that there are n equations. If the number of equations be greater than the number of unknowns, the values of the unknowns, in general, cannot be found to satisfy the equations. When they do satisfy all the given equations only n of the equations are independent, and the system of equations is then said to be *consistent*.

N.B.—In the following general examples, R_1, R_2, R_3, C_1, C_2, C_3 will be used to denote the first, second, and third rows, and the first, second, and third columns, respectively. One of the chief aims, when simplifying a determinant, by use of Section 8, is to get the terms of one row or column the same if possible, and it is always advisable to try adding all three columns (or rows) to see if the results are the same.

Example 5.—Evaluate (i) $\Delta \equiv \begin{vmatrix} 3 & 5 & 7 \\ 11 & 9 & 13 \\ 15 & 17 & 19 \end{vmatrix}$, (ii) $\Delta \equiv \begin{vmatrix} 13 & 3 & 23 \\ 30 & 7 & 53 \\ 39 & 9 & 70 \end{vmatrix}$.

(i) $$\Delta = \begin{vmatrix} 3 & 2 & 2 \\ 11 & -2 & 4 \\ 15 & 2 & 2 \end{vmatrix} \qquad (C_3 - C_2 \text{ and } C_2 - C_1)$$

$$= \begin{vmatrix} 3 & 2 & 2 \\ 5 & -6 & 0 \\ 12 & 0 & 0 \end{vmatrix} \qquad (R_2 - 2R_1 \text{ and } R_3 - R_1)$$

$$= 2\begin{vmatrix} 5 & -6 \\ 12 & 0 \end{vmatrix} = 2(-1)(-6)(12) = 144.$$

(ii)
$$\Delta = \begin{vmatrix} 13 & 3 & 23 \\ 30 & 7 & 53 \\ 0 & 0 & 1 \end{vmatrix} \qquad (R_3 - 3R_1)$$
$$= \begin{vmatrix} 0 & 0 & 1 \\ 13 & 3 & 23 \\ 30 & 7 & 53 \end{vmatrix} \qquad \text{(interchanging rows twice)}$$
$$= \begin{vmatrix} 13 & 3 \\ 30 & 7 \end{vmatrix} = 91 - 90 = 1.$$

Example 6.—Evaluate $\Delta \equiv \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$.

$$\Delta = \begin{vmatrix} 2b & 2a & 0 \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \qquad (R_1 + \{R_2 - R_3\})$$
$$= 2\begin{vmatrix} b & a & 0 \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 2\begin{vmatrix} b & a & 0 \\ 0 & c & b \\ c & c & a+b \end{vmatrix} \qquad (R_2 - R_1)$$
$$= 2\begin{vmatrix} b & a & 0 \\ 0 & c & b \\ c & 0 & a \end{vmatrix} \qquad (R_3 - R_2)$$
$$= 2(abc + abc) = 4abc.$$

Example 7.—Prove the identity $\begin{vmatrix} a^2 & a & bc \\ b^2 & b & ca \\ c^2 & c & ab \end{vmatrix} = \begin{vmatrix} a^3 & a^2 & 1 \\ b^3 & b^2 & 1 \\ c^3 & c^2 & 1 \end{vmatrix}$.

$$\begin{vmatrix} a^2 & a & bc \\ b^2 & b & ca \\ c^2 & c & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a^3 & a^2 & abc \\ b^3 & b^2 & abc \\ c^3 & c^2 & abc \end{vmatrix} \qquad (R_1 \times a,\ R_2 \times b,\ R_3 \times c, \text{ and divide by } abc)$$
$$= \frac{abc}{abc}\begin{vmatrix} a^3 & a^2 & 1 \\ b^3 & b^2 & 1 \\ c^3 & c^2 & 1 \end{vmatrix} = \begin{vmatrix} a^3 & a^2 & 1 \\ b^3 & b^2 & 1 \\ c^3 & c^2 & 1 \end{vmatrix}.$$

Example 8.—Solve the equation $\begin{vmatrix} 4x+5 & 4x+7 & 4x+9 \\ 4x+9 & 4x+5 & 4x+7 \\ 4x+7 & 4x+9 & 4x+5 \end{vmatrix} = 0.$

Now
$$\begin{vmatrix} 4x+5 & 4x+7 & 4x+9 \\ 4x+9 & 4x+5 & 4x+7 \\ 4x+7 & 4x+9 & 4x+5 \end{vmatrix} = \begin{vmatrix} 12x+21 & 4x+7 & 4x+9 \\ 12x+21 & 4x+5 & 4x+7 \\ 12x+21 & 4x+9 & 4x+5 \end{vmatrix} \qquad (C_1 + \{C_2 + C_3\})$$
$$= (12x+21)\begin{vmatrix} 1 & 4x+7 & 4x+9 \\ 1 & 4x+5 & 4x+7 \\ 1 & 4x+9 & 4x+5 \end{vmatrix}$$
$$= (12x+21)\begin{vmatrix} 1 & 4x+7 & 4x+9 \\ 0 & -2 & -2 \\ 0 & 2 & -4 \end{vmatrix} \qquad (R_2 - R_1 \text{ and } R_3 - R_1)$$
$$= (12x+21)\begin{vmatrix} -2 & -2 \\ 2 & -4 \end{vmatrix} = 12(12x+21) = 36(4x+7),$$

Thus the equation reduces to $4x + 7 = 0$ and the solution is $x = -7/4$

6. (i) Solve completely the equation

$$\begin{vmatrix} 3x & 2 & 3x-4 \\ 6x-2 & 9x-2 & 4 \\ 6x+3 & 9x & 5 \end{vmatrix} = 0.$$

(ii) Show that the equations

$$2x + 3y = 4,\ \ 3x + \lambda y = -1,\ \ \lambda x - 2y = c$$

are consistent for real values of λ if $4c^2 - 156c - 439 \geqslant 0$.

7. (i) Solve the following equation in x:

$$\begin{vmatrix} -1 & 2 & 2 \\ x^2+ax & -ax & ax+a^2 \\ x^2+ax & ax+a^2 & -ax \end{vmatrix} = 0.$$

(ii) Show that a and $(a + b + c)$ are factors of the determinant

$$\begin{vmatrix} (b+c)^2 & b^2 & c^2 \\ a^2 & (c+a)^2 & c^2 \\ a^2 & b^2 & (a+b)^2 \end{vmatrix}.$$

Express the determinant as a product of factors.

8. (i) Show that $x = 3$ is a root of the equation

$$\begin{vmatrix} x & -6 & -1 \\ 3 & -2x & x-4 \\ -2 & 3x & x-2 \end{vmatrix} = 0,$$

and solve it completely.

(ii) Prove that
$$\begin{vmatrix} (y-z)^3 & (z-x)^3 & (x-y)^3 \\ (y-z)^2 & (z-x)^2 & (x-y)^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (-2x + y + z)(x - 2y + z)(x + y - 2z)(x^2 + y^2 + z^2 - yz - zx - xy).$$

9. (i) Show that
$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$$

(ii) By multiplying the second column by b, the third column by c, and subtracting the elements of one column from the other (or by any other method), show that a is a factor of the determinant

$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}.$$

Evaluate the determinant.

10. (i) Show that $\sin\alpha$, $\cos\alpha$, $(\sin\alpha - \cos\alpha)$ are factors of the determinant

$$\begin{vmatrix} \cos\alpha & \sin 2\alpha & \cos^2\alpha \\ \sin\alpha & \sin 2\alpha & \sin^2\alpha \\ \sin\alpha & \sin^2\alpha & \cos^2\alpha \end{vmatrix},$$

and find the remaining factors.

(ii) Find the values of λ for which the following equations are consistent:

$$\begin{aligned} 3x + \lambda y &= 5, \\ \lambda x - 3y &= -4, \\ 3x - y &= -1. \end{aligned}$$

Solve these equations for these values of λ.

11. (i) Solve the equation $\begin{vmatrix} x+1 & x+2 & 3 \\ 2 & x+3 & x+1 \\ x+3 & 1 & x+2 \end{vmatrix} = 0.$

(ii) Prove that $\begin{vmatrix} (x+1)(x+2) & x+2 & 1 \\ (x+2)(x+3) & x+3 & 1 \\ (x+3)(x+4) & x+4 & 1 \end{vmatrix} = -2.$

12. Prove that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b).$

If a, b, c have all different values, and

$$\begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix} = 0,$$

prove that $abc = 1$.

13. (i) Express the determinant $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ as the product of simple factors.

(ii) Solve, by determinants, the equations

$$\begin{aligned} 4x - 3y + 2z + 7 &= 0, \\ 6x + 2y - 3z - 33 &= 0, \\ 2x - 4y - z + 3 &= 0. \end{aligned}$$

14. Express by means of a determinant the condition that the three equations

$$\begin{aligned} (3+\lambda)x + (2+2\lambda)y + (\lambda - 2) &= 0, \\ (2\lambda - 3)x + (2-\lambda)y + 3 &= 0, \\ 3x + 7y - 1 &= 0 \end{aligned}$$

should be consistent. Hence determine the possible values of λ.

CHAPTER IX

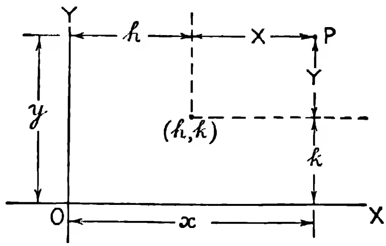
Plane Co-ordinate Geometry—The Straight Line, Circle, and Parabola

The Straight Line

1. Transferring the origin.

Given the curve $y = f(x)$, and (X, Y) the co-ordinates of any point P, which was originally the point (x, y) before the origin was transferred to the point (h, k), the direction of the axes being the same, then, from the diagram, it is seen that

$$x = X + h,$$
$$y = Y + k.$$



Thus, when the origin is transferred to the point (h, k), the equation of the curve becomes

$$Y + k = f(X + h),$$

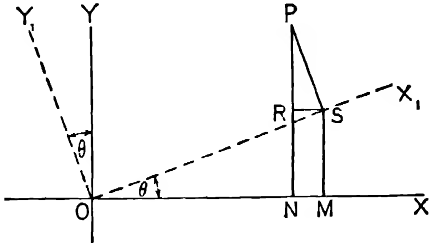
where (X, Y) are the new co-ordinates of the point which was the point (x, y) with respect to the original axes.

2. Turning the axes through an angle.

Consider the axes OX and OY turned through an angle θ in a positive direction, the origin O being kept the same, and let the new axes be OX_1 and OY_1 respectively.

Let P be the point (x, y) with respect to the axes OX, OY, and

(X, Y) with respect to the new axes OX_1, OY_1. PN, PS are the perpendiculars from P on OX and OX_1 respectively. Therefore $\angle SPN = \theta$ SR and SM are the perpendiculars from S on PN and OX respectively



$$
\begin{aligned}
\text{From the diagram, } x &= \text{ON} = \text{OM} - \text{NM} = \text{OS} \cos\theta - \text{RS} \\
&= \text{OS} \cos\theta - \text{PS} \sin\theta \\
&= X \cos\theta - Y \sin\theta. \\
y &= \text{PN} = \text{PR} + \text{RN} = \text{PR} + \text{SM} \\
&= \text{PS} \cos\theta + \text{OS} \sin\theta \\
&= X \sin\theta + Y \cos\theta.
\end{aligned}
$$

Thus the equation of the curve $y = f(x)$, with respect to the axes OX, OY, becomes, with respect to the axes OX_1, OY_1,

$$X \sin\theta + Y \cos\theta = f(X \cos\theta - Y \sin\theta).$$

3. Theorem.—To find the point of intersection of the straight lines

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0.$$

Using determinants, the point of intersection is given by

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

4. Theorem.—To find the condition that the three lines represented by

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0,$$
$$a_3x + b_3y + c_3 = 0$$

shall be concurrent.

If the given lines are to be concurrent, unique values for x and y can be found to satisfy the given equations, i.e. the given equations must be consistent, and the condition for this is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

5. Theorem.—To find the equation of any straight line through the point of intersection of two given straight lines.

Let
$$a_1x + b_1y + c_1 = 0, \quad \ldots \ldots \quad (1)$$
$$a_2x + b_2y + c_2 = 0 \quad \ldots \ldots \quad (2)$$

be the equations of the two given lines.

If (x_1, y_1) be their point of intersection, then

$$a_1x_1 + b_1y_1 + c_1 = 0, \quad \ldots \ldots \quad (3)$$
$$a_2x_1 + b_2y_1 + c_2 = 0. \quad \ldots \ldots \quad (4)$$

Now consider the line (equation linear in x and y) represented by

$$a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0, \quad \ldots \quad (5)$$

where λ is any arbitrary constant.

Substituting $x = x_1$, $y = y_1$ in the left-hand side of (5), it becomes

$$a_1x_1 + b_1y_1 + c_1 + \lambda(a_2x_1 + b_2y_1 + c_2).$$

Using (3) and (4) the value of this is zero, i.e. (x_1, y_1) lies on the line represented by equation (5).

Thus any line passing through the point of intersection of the lines (1) and (2) will be

$$\boldsymbol{a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0}$$

for all values of λ.

The value of λ can be determined in any particular case by considering the data given in the problem.

Example 1.—Find the equation of the straight line passing through the point of intersection of the lines $2x + 3y + 4 = 0$, $5x + 2y - 3 = 0$, and perpendicular to the line $x - y = 0$.

The equation of any straight line through the point of intersection of the first two lines is

$$2x + 3y + 4 + \lambda(5x + 2y - 3) = 0, \quad \ldots \ldots \quad (\text{i})$$

and the slope of this line is $-(2 + 5\lambda)/(3 + 2\lambda)$.

Since the line (i) is perpendicular to $x - y = 0$, its slope is -1.

$$\therefore\ -(2 + 5\lambda)/(3 + 2\lambda) = -1.$$

$$\therefore\ 2 + 5\lambda = 3 + 2\lambda, \text{ i.e. } \lambda = \tfrac{1}{3}.$$

Therefore equation of required line is

$$2x + 3y + 4 + \tfrac{1}{3}(5x + 2y - 3) = 0,$$

i.e.
$$11x + 11y + 9 = 0.$$

6. Theorem.—To find the angle θ between the two straight lines represented by the equation

$$ax^2 + 2bxy + cy^2 = 0.$$

Let the two straight lines represented by the given equation be

$$y = m_1x, \quad \text{. . . . . (6)}$$

$$y = m_2x. \quad \text{. . . . . (7)}$$

Using the theory of quadratic equations, since the given equation can be represented by

$$(y - m_1x)(y - m_2x) = 0,$$

it follows that
$$m_1 + m_2 = -2b/c, \quad \text{. . . . . (8)}$$

$$m_1m_2 = a/c. \quad \text{. . . . . (9)}$$

$$\therefore\ \tan\theta = \pm\frac{m_1 - m_2}{1 + m_1m_2} = \pm\frac{\sqrt{\{(m_1 + m_2)^2 - 4m_1m_2\}}}{1 + m_1m_2}$$

$$= \pm\sqrt{\frac{4b^2}{c^2} - \frac{4a}{c}}\bigg/\left(1 + \frac{a}{c}\right)$$

$$= \pm\, 2\sqrt{b^2 - ac}/(a + c).$$

$$\therefore\ \theta = \tan^{-1} \pm\frac{2\sqrt{b^2 - ac}}{a + c}.$$

From this result it can be seen that the lines will be

(i) real, if $b^2 > ac$;

(ii) coincident, if $b^2 = ac$;

(iii) imaginary, if $b^2 < ac$;

(iv) perpendicular, if $a + c = 0$.

7. Theorem.—To find the condition that the general equation of the second degree in x and y shall represent two straight lines.

In the standard form, the general equation of the second degree in x and y is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad . \quad . \quad (10)$$

This can be written

$$ax^2 + 2x(hy + g) + by^2 + 2fy + c = 0. \quad . \quad . \quad (11)$$

Solving (11) as a quadratic in x,

$$\begin{aligned} x &= \{-(hy + g) \pm \sqrt{[(hy + g)^2 - a(by^2 + 2fy + c)]}\}/a \\ &= \{-(hy + g) \pm \sqrt{[y^2(h^2 - ab) + 2y(hg - af) + (g^2 - ac)]}\}/a. \quad (12) \end{aligned}$$

In order that (10) shall represent two straight lines, the quantity under the root sign in (12) must be a complete square.

Thus the necessary condition is

$$(hg - af)^2 = (h^2 - ab)(g^2 - ac),$$

which reduces to

$$\boldsymbol{abc + 2fgh - af^2 - bg^2 - ch^2 = 0.}$$

This result can be put in the determinant form

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

8. Theorem.—To find the equation for the pair of straight lines joining the origin O to the points of intersection of the straight line $lx + my = 1$ and the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Making the second equation homogeneous and of the second degree in x and y, by means of the first equation,

$$ax^2 + 2hxy + by^2 + (2gx + 2fy)(lx + my) + c(lx + my)^2 = 0. \quad (13)$$

Since equation (13) is homogeneous in x and y and of the second degree it represents two straight lines through the origin. It is also easily seen that, if (x_1, y_1) be a point of intersection of the line and the curve, it will also lie on one of the lines represented by equation (13).

Thus equation (13) must be the required equation.

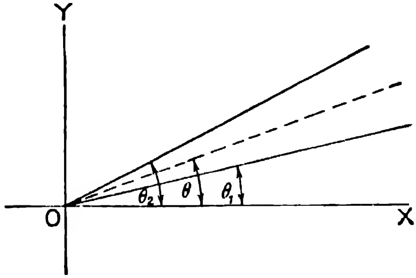
9. Theorem.—To find the equation of the pair of straight lines bisecting the angles between the two lines given by

$$ax^2 + 2hxy + by^2 = 0.$$

Let the lines represented by

$$ax^2 + 2hxy + by^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (14)$$

have slopes m_1 and m_2, and make angles θ_1 and θ_2 with OX. Let one of the bisectors make an angle θ with OX.



From the diagram,

$$\theta = \frac{\theta_2 - \theta_1}{2} + \theta_1 \text{ or } \frac{\pi}{2} + \frac{\theta_2 - \theta_1}{2} + \theta_1$$

$$= \frac{\theta_1 + \theta_2}{2} \text{ or } \frac{\pi}{2} + \frac{\theta_1 + \theta_2}{2}.$$

$$\therefore\ 2\theta = \theta_1 + \theta_2 \text{ or } \pi + (\theta_1 + \theta_2).$$

In either case

$$\tan 2\theta = \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$

$$= \frac{m_1 + m_2}{1 - m_1 m_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

From equation (14),

$$m_1 + m_2 = -\frac{2h}{b},\ m_1 m_2 = \frac{a}{b}.$$

Also if (x, y) be any point on one of the bisectors,

$$y/x = \tan\theta.$$

Using these in equation (15), i.e. $\dfrac{2\tan\theta}{1 - \tan^2\theta} = \dfrac{m_1 + m_2}{1 - m_1 m_2}$,

$$\frac{2xy}{x^2 - y^2} = -\frac{2h}{b - a},$$

i.e. the required equation is $\dfrac{x^2 - y^2}{a - b} = \dfrac{xy}{h}$.

10. Polar equation of a straight line.

If OX be the initial line, and OP be a line making an angle θ with OX, where OP is of length r, it has been shown that, if $P \equiv (x, y)$, then

$$x = r \cos\theta \text{ and } y = r \sin\theta.$$

Considering the perpendicular form of the equation of a straight line

$$x \cos\alpha + y \sin\alpha = p,$$

and using the above, it becomes

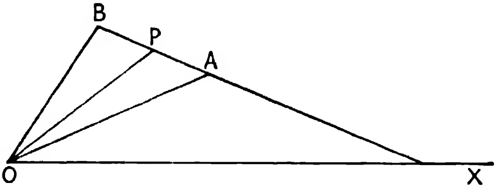
$$r(\cos\theta \cos\alpha + \sin\theta \sin\alpha) = p,$$

i.e.
$$\boldsymbol{r \cos(\theta - \alpha) = p},$$

where α is the angle that the perpendicular from the origin on the line makes with OX, and p is the length of this perpendicular.

11. Theorem.—To find the polar equation of the straight line passing through the points (r_1, θ_1), (r_2, θ_2).

Let $A \equiv (r_1, \theta_1)$, $B \equiv (r_2, \theta_2)$, and let $P \equiv (r, \theta)$ be any point on the join of AB.



From the diagram

$$2\triangle AOP + 2\triangle POB - 2\triangle AOB = 0,$$

i.e. $$\boldsymbol{rr_1 \sin(\theta - \theta_1) + rr_2 \sin(\theta_2 - \theta) - r_1r_2 \sin(\theta_2 - \theta_1) = 0},$$

which is the required equation.

Example 2 (L.U.).—Show that the bisectors of the angles between the lines $ax^2 + 2hxy + by^2 = 0$ are given by the equation $hx^2 - (a - b)xy - hy^2 = 0$.

Show that the equation

$$3x^2 - 4xy - 4y^2 + 14x + 12y - 5 = 0$$

represents two straight lines, and find the combined equation of the bisectors of the angles between them.

The first part of the question has been proved as bookwork.

$$3x^2 - 4xy - 4y^2 + 14x + 12y - 5 = 0. \quad \ldots \quad \text{(i)}$$

The equation (i) could be proved to represent two straight lines by using the fact that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will represent two straight lines, if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, but it is better in this case to show that the left-hand side of equation (i) can be factorized, as the point of intersection of the two lines will be required for the last part of the question.

Now $$3x^2 - 4xy - 4y^2 = (3x + 2y)(x - 2y).$$

Thus if the left-hand side of (i) can be factorized, the result will be

$$(3x + 2y + a)(x - 2y + b),$$

the expansion of which is

$$3x^2 - 4xy - 4y^2 + x(a + 3b) + y(2b - 2a) + ab,$$

and which must be identical with the left-hand side of (i).

Comparing coefficients

of x, $$a + 3b = 14, \qquad \text{(ii)}$$

of y, $$2b - 2a = 12, \quad \text{i.e. } b - a = 6, \qquad \text{(iii)}$$

of unity, $$ab = -5. \qquad \text{(iv)}$$

From (ii) and (iii), $b = 5$ and $a = -1$,

which satisfy (iv).

Therefore the equation (i) represents the two lines

$$3x + 2y - 1 = 0, \qquad \text{(v)}$$
$$x - 2y + 5 = 0. \qquad \text{(vi)}$$

The lines (v) and (vi) intersect in the point $(-1, 2)$. Transferring the origin to this point, the equation of the lines (i) will reduce to

$$3x^2 - 4xy - 4y^2 = 0.$$

Using the first part of the question with $(-1, 2)$ as the origin, the equation of the bisectors (using $a = 3$, $h = -2$, $b = -4$) will be

$$-2x^2 - \{3 - (-4)\}xy + 2y^2 = 0,$$

i.e. $$2x^2 + 7xy - 2y^2 = 0;$$

and reverting to the original origin, the required equation is

$$2(x + 1)^2 + 7(x + 1)(y - 2) - 2(y - 2)^2 = 0,$$

i.e. $$2x^2 + 7xy - 2y^2 - 10x + 15y - 20 = 0.$$

Example 3 (L.U.).—Show that the equation $ax^2 + 2hxy + by^2 = 0$ represents two straight lines, and that the equation of the lines through the origin perpendicular to these is $bx^2 - 2hxy + ay^2 = 0$.

Lines through a point A, perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$, cut the x-axis at the points L and M, and the y-axis at the points P and Q. The mid-point of LM is N; the mid-point of PQ is R; the mid-point of NR is B; and O is the origin. If O, A, B are collinear, show that A lies on the lines $bx^2 = ay^2$.

Since the equation $$ax^2 + 2hxy + by^2 = 0 \quad \ldots \ldots \quad \text{(i)}$$

can always be factorized into

$$(a_1x + b_1y)(a_2x + b_2y) = 0, \quad \ldots \ldots \quad \text{(ii)}$$

where a_1, b_1, a_2, b_2 may be real or complex, it follows that the equation (i) must always represent two straight lines, which may be real, coincident, or imaginary.

Comparing equations (i) and (ii),

$$\left.\begin{aligned} a_1a_2 &= a \\ a_1b_2 + a_2b_1 &= 2h \\ b_1b_2 &= b \end{aligned}\right\}. \quad \ldots \ldots \quad \text{(iii)}$$

Lines perpendicular to the lines represented by (ii) which pass through the origin will have as their equations

$$b_1x - a_1y = 0,$$
$$b_2x - a_2y = 0,$$

and their combined equation is

$$(b_1x - a_1y)(b_2x - a_2y) = 0,$$

i.e. $$b_1b_2x^2 - (a_1b_2 + a_2b_1)xy + a_1a_2y^2 = 0.$$

Hence, using (iii), the combined equation is

$$bx^2 - 2hyx + ay^2 = 0.$$

Let A be the point $(\bar{x}, \bar{y})$.

The equation of the lines through O perpendicular to the lines given by equation (i) is

$$bx^2 - 2hxy + ay^2 = 0.$$

The equation of the lines through A perpendicular to the lines (i) will be

$$b(x - \bar{x})^2 - 2h(x - \bar{x})(y - \bar{y}) + a(y - \bar{y})^2 = 0. \quad \ldots \quad \text{(iv)}$$

At L and M, $y = 0$,

i.e. $$b(x - \bar{x})^2 + 2h\bar{y}(x - \bar{x}) + a\bar{y}^2 = 0,$$

i.e. $$bx^2 - 2x(b\bar{x} - h\bar{y}) + (b\bar{x}^2 - 2h\bar{x}\bar{y} + a\bar{y}^2) = 0.$$

If x_1 and x_2 be the roots of this quadratic (i.e. the values of x at L and M),

$$x_1 + x_2 = 2(b\bar{x} - h\bar{y})/b.$$

The x-co-ordinate at N is

$$(x_1 + x_2)/2 = (b\bar{x} - h\bar{y})/b.$$

$$\therefore \text{N} = \left(\frac{b\bar{x} - h\bar{y}}{b}, 0\right).$$

At P and Q, $x = 0$,

i.e. $$b\bar{x}^2 + 2h\bar{x}(y - \bar{y}) + a(y - \bar{y})^2 = 0,$$

i.e. $$ay^2 - 2y(a\bar{y} - h\bar{x}) + a\bar{y}^2 - 2h\bar{x}\bar{y} + b\bar{x}^2 = 0.$$

From which, as before, $R \equiv \left(0, \dfrac{a\bar{y} - h\bar{x}}{a}\right)$.

Hence the mid-point B of NR is the point

$$\left(\frac{b\bar{x} - h\bar{y}}{2b}, \frac{a\bar{y} - h\bar{x}}{2a}\right).$$

If O, A, B be collinear, the slope of OA, i.e. $\bar{y}/\bar{x}$, must equal the slope of OB, i.e. $\dfrac{a\bar{y} - h\bar{x}}{2a} \div \dfrac{b\bar{x} - h\bar{y}}{2b}$,

$$\therefore \frac{\bar{y}}{\bar{x}} = \frac{b(a\bar{y} - h\bar{x})}{a(b\bar{x} - h\bar{y})},$$

i.e. $$a\bar{y}(b\bar{x} - h\bar{y}) = b\bar{x}(a\bar{y} - h\bar{x}),$$

i.e. $$ab\bar{x}\bar{y} - ah\bar{y}^2 = ab\bar{x}\bar{y} - bh\bar{x}^2,$$

i.e. $$a\bar{y}^2 = b\bar{x}^2.$$

Changing to running co-ordinates, the locus of A is the lines represented by

$$bx^2 = ay^2.$$

THE CIRCLE

12. Theorem.—To find the condition that the point (x_1, y_1) lies on, inside or outside the circle $x^2 + y^2 = r^2$.

Let P be the point (x_1, y_1).

If P lie on the circle, $x_1^2 + y_1^2 = r^2$.

If P lie inside the circle, $OP^2 < r^2$,

$$\therefore x_1^2 + y_1^2 < r^2.$$

If P lie outside the circle, then

$$OP^2 > r^2,$$

i.e. $$x_1^2 + y_1^2 > r^2.$$

Hence the point P lies within, on, or outside the circle $x^2 + y^2 = r^2$, according as $x_1^2 + y_1^2 <$ or $=$ or $> r^2$.

13. Theorem.—To prove that two tangents can be drawn to a circle from any point, and these two tangents will be real, coincident, or imaginary, according as the point lies outside, on, or inside the circle.

Taking the origin as the centre of the circle, the equation of a circle of radius r will be

$$x^2 + y^2 = r^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Let (h, k) be the point from which the two tangents are drawn to the circle, and let (x_1, y_1) be a point of contact of a tangent from (h, k).

Then the equation of the tangent will be

$$xx_1 + yy_1 = r^2. \quad \ldots \ldots \quad (17)$$

Since the line (17) passes through the point (h, k),

$$hx_1 + ky_1 = r^2. \quad \ldots \ldots \quad (18)$$

Since the point (x_1, y_1) lies on the circle (16),

$$x_1^2 + y_1^2 = r^2. \quad \ldots \ldots \quad (19)$$

Eliminating y_1 between equations (18) and (19),

$$x_1^2 + \left(\frac{r^2 - hx_1}{k}\right)^2 = r^2,$$

i.e.
$$x_1^2(k^2 + h^2) - 2x_1r^2h + r^2(r^2 - k^2) = 0. \quad \ldots \quad (20)$$

The equation (20) gives the abscissæ of the points of contact of the tangents from the point (h, k) to the circle (16) and, since it is a quadratic in x_1, there are two points of contact,

i.e. *two tangents can be drawn to a circle from any point.*

These points of contact will be real, coincident, or imaginary points, according as the roots of (20) are real, coincident, or imaginary.

Hence the two tangents will be real, coincident, or imaginary, according as

$$r^4h^2 > \text{ or } = \text{ or } < r^2(k^2 + h^2)(r^2 - k^2),$$

i.e. according as

$$r^2h^2 > \text{ or } = \text{ or } < r^2k^2 + r^2h^2 - k^4 - k^2h^2,$$

i.e. according as

$$k^2 + h^2 > \text{ or } = \text{ or } < r^2 \qquad (\text{since } k^2 \neq 0),$$

i.e. according as the *point (h, k) lies outside, on, or inside the given circle.*

14. Theorem.—Two tangents are drawn to a circle from any point (h, k). To find the equation of the straight line (chord of contact) joining the points of contact (real, coincident, or imaginary).

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0. \quad \ldots \ldots \quad (21)$$

The equations of the tangents at (x_1, y_1), (x_2, y_2) are respectively

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0, \quad . \quad (22)$$

$$xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c = 0. \quad . \quad (23)$$

Since the point (h, k) lies on lines (22) and (23),

$$hx_1 + ky_1 + g(h + x_1) + f(k + y_1) + c = 0, \quad . \quad (24)$$

$$hx_2 + ky_2 + g(h + x_2) + f(k + y_2) + c = 0. \quad . \quad (25)$$

But (24) and (25) are the conditions that the points (x_1, y_1), (x_1, y_2) shall lie on the line

$$hx + ky + g(x + h) + f(y + k) + c = 0. \quad . \quad . \quad (26)$$

Hence equation (26) is the equation of the line joining the points of contact of the tangents from the point (h, k) to the given circle. This line (or chord of contact) is known as the *polar* of the point (h, k) with respect to the given circle, and the point (h, k) is known as the *pole* of this line.

Thus the polar of (h, k) with respect to the circle (21) is given by

$$hx + ky + g(x + h) + f(y + k) + c = 0.$$

It is to be noted that the polar of (h, k) is real, whether the point (h, k) lies outside, on, or inside the given circle, whilst the tangents from (h, k) to the circle are only real if the point (h, k) lies outside or on the circle.

In the special case when the origin is the centre, and the equation of the circle is $x^2 + y^2 = a^2$, the polar of (h, k) is $hx + ky = a^2$.

N.B.—The form of the equation of the polar is the same as that of the tangent at a point on the circle.

15. Theorem.—To prove that, if the polar of P with respect to a given circle passes through Q, then the polar of Q will pass through P.

Let $\qquad \mathrm{P} \equiv (x_1, y_1),\ \mathrm{Q} \equiv (x_2, y_2),$

and the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0. \quad . \quad . \quad . \quad . \quad (27)$$

The polar of P with respect to the circle (27) is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0. \quad . \quad (28)$$

Since this line passes through Q,

$$x_2x_1 + y_2y_1 + g(x_2 + x_1) + f(y_2 + y_1) + c = 0. \quad . \quad (29)$$

But equation (29) is the condition that the polar of (x_2, y_2) shall pass through (x_1, y_1), since the polar of (x_2, y_2) with respect to circle (27) is

$$xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c = 0.$$

Hence, if the polar of P *passes through* Q, *the polar of* Q *will pass through* P.

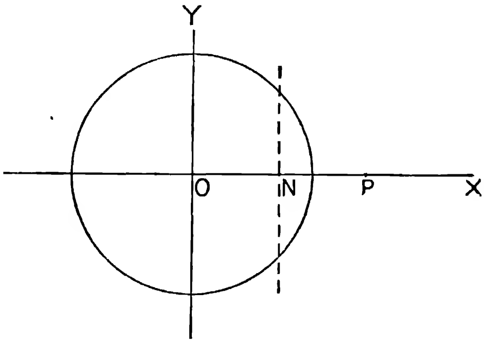
16. Theorem.—If the polars of P and Q with respect to any circle pass through R, then the polar of R is the line PQ.

Since the polar of P passes through R, the polar of R must pass through P. Similarly, the polar of R must pass through Q.

Hence *the polar of* R *is the line* PQ.

17. Theorem.—To prove that the polar of P with respect to a circle centre O is perpendicular to OP and, if N be the point in which the polar of P cuts OP, then ON . OP $= a^2$, where the circle has radius a.

N.B.—In all co-ordinate geometry, if choice of origin and axes be permitted, they must be chosen so as to give the simplest solution.



Taking O as the origin of co-ordinates and OP as the x-axis, the equation of the circle will be

$$x^2 + y^2 = a^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and the point P $\equiv (x_1, 0)$. The polar of P with respect to the circle (30) is $xx_1 = a^2$. Hence *the polar of* P *is perpendicular to* OP *and* ON . OP $= a^2$.

Note.—This result gives a method of constructing the polar of any point P with respect to a given circle.

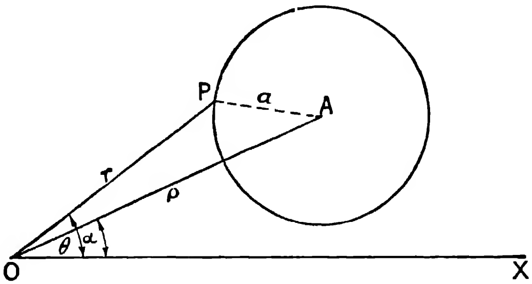
18. Theorem.—To find the polar equation of a circle.

Let O be the origin, A the centre of the circle of radius a, and OX the initial line making an angle α with OA. Let OA $= \rho$, and P $\equiv$ (r, θ) be any point on the circle. Using the cosine rule on ΔAOP,

$$a^2 = \rho^2 + r^2 - 2r\rho\cos(\theta - \alpha).$$

Hence the required equation of the circle is

$$\boldsymbol{a^2 = \rho^2 + r^2 - 2r\rho\cos(\theta - \alpha)}. \quad . \quad . \quad . \quad (31)$$



If the point O lie on the circle, then $\rho = a$, and the equation becomes, on dividing through by r,

$$\boldsymbol{r = 2a\cos(\theta - \alpha)}.$$

If, in addition, OX be a diameter of the circle, then $\alpha = 0$, and the equation is

$$\boldsymbol{r = 2a\cos\theta}.$$

From equation (31) considered as a quadratic in r, it can be seen that the product of the roots (r_1 and r_2) is equal to $\rho^2 - a^2$. Hence the product of the segments of a secant of a circle is constant and equal to $\rho^2 - a^2$.

19. Theorem.—If $S = 0$ and $S_1 = 0$ represent two circles, where $S \equiv x^2 + y^2 + 2gx + 2fy + c$, and $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1$, to find what the equation $S - S_1 = 0$ represents.

Now a point of intersection of $S = 0$ and $S_1 = 0$ will satisfy both the equations $S = 0$ and $S_1 = 0$, and will therefore satisfy the equation $S - S_1 = 0$.

Thus the curve given by $S - S_1 = 0$ passes through the points of intersection of the two given circles.

Also $S - S_1 = 0$ reduces to

$$2x(g - g_1) + 2y(f - f_1) + c - c_1 = 0,$$

which is the equation of a straight line.

Hence the curve $S - S_1 = 0$ represents the straight line passing through the points of intersection (real, coincident, or imaginary) of the two circles. This straight line is known as the *radical axis* of the two circles, and is the common chord when the circles intersect in real points.

Putting the equation in the form $S = S_1$, it can be seen that the tangents from any point on the radical axis of the two circles are equal, since S gives the value of the square of the tangent from (x, y) to the circle $S = 0$. Thus the radical axis can be defined as

(i) the straight line passing through the points of intersection (real, coincident, or imaginary) of the two given circles, or

(ii) the locus of points from which the tangents to the two given circles are equal.

20. Theorem.—To prove that the radical axis of the two circles $S = 0$ and $S_1 = 0$ is perpendicular to their line of centres (notation of Section 19).

The radical axis of $S = 0$ and $S_1 = 0$ has the equation

$$2x(g - g_1) + 2y(f - f_1) + c - c_1 = 0,$$

and its slope is

$$-(g - g_1)/(f - f_1).$$

The centre of $S = 0$ is $(-g, -f)$, and the centre of $S_1 = 0$ is $(-g_1, -f_1)$, therefore the slope of the line of centres is $(f - f_1)/(g - g_1)$. The product of the two slopes is -1, and hence *the radical axis is perpendicular to the line of centres.*

21. Theorem.—To prove that the three radical axes of three given circles taken in pairs are concurrent.

Let

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0, \quad . \quad . \quad (32)$$
$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0, \quad . \quad . \quad (33)$$
$$S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0 \quad . \quad . \quad (34)$$

be the three given circles.

Then the radical axis of circles (32) and (33) is

$$S_1 - S_2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

and the radical axis of circles (33) and (34) is

$$S_2 - S_3 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Taking (35) + (36), an equation of a line passing through the point of intersection of the lines (35) and (36) is obtained, and this equation is $S_1 - S_3 = 0$, which is the equation of the radical axis of circles (32) and (34).

Thus *the three radical axes are concurrent.* The point in which these three radical axes meet is known as the *radical centre.*

22. Theorem.—To find the equation of a system of circles every pair of which has the same radical axis.

Choose as the y-axis the common radical axis of the system of circles, and as the x-axis the common line of centres.

Two circles of the system will be

$$x^2 + y^2 + 2g_1x + c_1 = 0,$$

and

$$x^2 + y^2 + 2g_2x + c_2 = 0.$$

Subtracting these equations, the radical axis is

$$2(g_1 - g_2)x + c_1 - c_2 = 0,$$

which gives $c_1 = c_2 = c$ (say) since the radical axis is $x = 0$.

Thus the required equation is

$$\boldsymbol{x^2 + y^2 + 2gx + c = 0}, \quad . \quad . \quad . \quad . \quad (37)$$

where c is constant and g arbitrary.

The family of circles having the same radical axis for any pair is known as a *coaxal system of circles.*

N.B.—From equation (37), the radical axis will cut the circle in real points if c is negative and imaginary points if c is positive.

The equation (37) can be written as

$$(x + g)^2 + y^2 = g^2 - c.$$

Point circles of the system, which are known as *limiting points* of the coaxal system, are obtained when $g^2 = c$, i.e. $g = \pm\sqrt{c}$.

These limiting points will be real if c is positive and imaginary if c is negative.

23. Theorem.—If $S = 0$ and $S_1 = 0$ be the equations of two circles (with usual notation for S and S_1), to prove that $S - \lambda S_1 = 0$ represents, for different values of the constant λ, all circles passing through the points common to $S = 0$ and $S_1 = 0$ (except $\lambda = 1$).

The equation $S - \lambda S_1 = 0$ reduces to

$$(1 - \lambda)(x^2 + y^2) + 2x(g - \lambda g_1) + 2y(f - \lambda f_1) + c - \lambda c_1 = 0,$$

which shows that the equation $S - \lambda S_1 = 0$ represents a circle except when $\lambda = 1$.

Also if (x_1, y_1) lie on both the circles $S = 0$ and $S_1 = 0$, it can be shown, as in the case of two straight lines, that the point (x_1, y_1) will lie on the curve represented by $S - \lambda S_1 = 0$.

Thus the equation $S - \lambda S_1 = 0$ represents a *circle passing through the points of intersection of the circles $S = 0$ and $S_1 = 0$.*

If the equation be put in the form $S = \lambda S_1$, it can be seen that the locus of points, such that the ratio of the tangents to two given circles is constant, will be a circle.

By a suitable choice of λ, the circle $S - \lambda S_1 = 0$ can be made to fulfil one further condition, such as passing through a given point, etc.

24. Definition.—Two curves are said to cut orthogonally (or "be orthogonal") if the tangents at a point of intersection are at right angles.

25. Theorem.—With the usual notation, to find the condition that the two circles $S = 0$ and $S_1 = 0$ shall be orthogonal.

From geometry, the radii at a point of intersection of the two orthogonal circles must be perpendicular, and, using Pythagoras' theorem, it follows that the sum of the squares of the radii is equal to the square on the line of centres.

The centre of $S = 0$ is $(-g, -f)$ and its radius is $\sqrt{(g^2 + f^2 - c)}$.
The centre of $S_1 = 0$ is $(-g_1, -f_1)$ and its radius is $\sqrt{(g_1{}^2 + f_1{}^2 - c_1)}$.
The square on the line of centres is

$$(g - g_1)^2 + (f - f_1)^2.$$

Thus the required condition is

$$(g^2 + f^2 - c) + (g_1{}^2 + f_1{}^2 - c_1) = (g - g_1)^2 + (f - f_1)^2,$$

which reduces to $\qquad 2gg_1 + 2ff_1 = c + c_1.$

Example 4 (L.U.).—Find the condition that the circles

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \text{ and } x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

should be orthogonal.

The circle $x^2 + y^2 + 2x - 4y - 11 = 0$ and the line $x - y + 1 = 0$ intersect at A and B. Find

(i) the equation of the circle on AB as diameter,

(ii) the equation of the circle through A and B orthogonal to the given circle.

The condition for the given circles to be orthogonal has been found in Section 25, and is in this case

$$2f_1f_2 + 2g_1g_2 = c_1 + c_2.$$

Now $$x^2 + y^2 + 2x - 4y - 11 = 0, \quad \ldots \quad \text{(i)}$$

and $$x - y + 1 = 0,$$

i.e. $$y = x + 1. \quad \ldots \quad \text{(ii)}$$

Substituting from (ii) in (i) for the points of intersection A and B,

$$x^2 + (x + 1)^2 + 2x - 4(x + 1) - 11 = 0,$$

i.e. $$2x^2 - 14 = 0, \quad \therefore\ x = \pm\sqrt{7} \text{ and } \therefore\ y = 1 \pm \sqrt{7},$$

i.e. $$A \equiv (\sqrt{7}, 1 + \sqrt{7}),\ B \equiv (-\sqrt{7}, 1 - \sqrt{7}).$$

The equation of the circle on (x_1, y_1), (x_2, y_2) as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

The equation of the circle on AB as diameter is

$$(x - \sqrt{7})(x + \sqrt{7}) + \{y - (1 + \sqrt{7})\}\{y - (1 - \sqrt{7})\} = 0,$$

i.e. $$x^2 - 7 + y^2 - 2y - 6 = 0,$$

i.e. $$x^2 + y^2 - 2y - 13 = 0.$$

Let the circle through A and B orthogonal to (i) be

$$x^2 + y^2 + 2gx + 2fy + c = 0. \quad \ldots \quad \text{(iii)}$$

Using the condition in the first part,

$$2g - 4f = c - 11. \quad \ldots \quad \text{(iv)}$$

Since A and B lie on the circle (iii),

$$7 + 8 + 2\sqrt{7} + 2g\sqrt{7} + 2f(1 + \sqrt{7}) + c = 0,$$

i.e. $$15 + 2\sqrt{7} + 2\sqrt{7}g + 2(1 + \sqrt{7})f + c = 0, \quad \ldots \quad \text{(v)}$$

and $$15 - 2\sqrt{7} - 2\sqrt{7}g + 2(1 - \sqrt{7})f + c = 0. \quad \ldots \quad \text{(vi)}$$

(v) + (vi) gives $$30 + 4f + 2c = 0,$$

i.e. $$15 + 2f + c = 0. \quad \ldots \quad \text{(vii)}$$

Eliminating c between (iv) and (vii),

$$2g - 2f = -26,$$

i.e.
$$g - f = -13. \quad \ldots \ldots \quad \text{(viii)}$$

(v) — (vi) gives $\quad 4\sqrt{7} + 4\sqrt{7}\,.\,g + 4\sqrt{7}\,.\,f = 0,$

$$\therefore\ 1 + g + f = 0. \quad \ldots \ldots \quad \text{(ix)}$$

From (viii) and (ix) $\quad g = -7,\ f = +6,$

therefore from (iv) $\quad c = -27.$

Hence required circle is

$$x^2 + y^2 - 14x + 12y - 27 = 0.$$

Example 5 (L.U.).—Show that, for constant c but varying λ, the equation $x^2 + y^2 + 2\lambda x + c^2 = 0$ represents a family of coaxal circles with real limiting points, and that for varying μ the equation $x^2 + y^2 + 2\mu y - c^2 = 0$ represents a family of coaxal circles orthogonal to the first family.

Find the polar of a point P with respect to any member of the μ family and show that for all values of μ these polars are concurrent at a point P_1. Show that PP_1 is a diameter of a circle of the λ system.

Consider the circles $\quad x^2 + y^2 + 2\lambda_1 x + c^2 = 0, \quad \ldots \ldots \quad$ (i)

$$x^2 + y^2 + 2\lambda_2 x + c^2 = 0. \quad \ldots \ldots \quad \text{(ii)}$$

(i) — (ii) gives for their points of intersection

$$(2\lambda_1 - 2\lambda_2)x = 0,$$

i.e.
$$x = 0, \text{ since } \lambda_1 \neq \lambda_2.$$

Hence the y-axis passes through their points of intersection and is their radical axis. The result is true for all values of λ_1 and λ_2, since c is constant,

therefore the equation $\quad x^2 + y^2 + 2\lambda x + c^2 = 0 \quad \ldots \ldots \quad$ (iii)

represents a system of circles with a common radical axis OY, i.e. *a coaxal system of circles.*

The equation (iii) can be written

$$(x + \lambda)^2 + y^2 = \lambda^2 - c^2.$$

The point circles (limiting points) of this coaxal system will be given when

$$\lambda^2 - c^2 = 0, \text{ i.e. } \lambda = \pm c.$$

λ is real and there are *real limiting points of the system.*

Similarly, $\quad x^2 + y^2 + 2\mu y - c^2 = 0 \quad \ldots \ldots \quad$ (iv)

represents a coaxal system of circles with common radical axis

$$y = 0, \text{ i.e. OX.}$$

Now the circles $\quad x^2 + y^2 + 2g_1 x + 2f_1 y + c_1 = 0,$

$$x^2 + y^2 + 2g_2 x + 2f_2 y + c_2 = 0$$

are orthogonal if $\quad 2g_1 g_2 + 2f_1 f_2 = c_1 + c_2.$

Circles (iii) and (iv) are orthogonal if

$$2\lambda \times 0 + 2\mu \times 0 = c^2 - c^2,$$

i.e. $$0 = 0, \text{ which is true.}$$

Therefore the *circles* (iii) *and* (iv) *are orthogonal* for all values of λ and μ.

Let P be the point (x_1, y_1).

The polar of P with respect to the circle (iv) is

$$xx_1 + yy_1 + \mu(y + y_1) - c^2 = 0,$$

i.e. using μ_1 and μ_2 to replace μ, the polars of P for the two circles (μ_1) and (μ_2) of the system will be

$$xx_1 + yy_1 + \mu_1(y + y_1) - c^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

$$xx_1 + yy_1 + \mu_2(y + y_1) - c^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

Solving (v) and (vi),

$$y = -y_1 \text{ and } x = (y_1^2 + c^2)/x_1,$$

i.e. since c, x_1, y_1 are fixed, the polars of P with respect to circles of the system μ pass through a fixed point $P_1 \equiv \left(\frac{y_1^2 + c^2}{x_1}, -y_1\right)$.

The circle on PP_1 as diameter is given by

$$(x - x_1)\left(x - \frac{y_1^2 + c^2}{x_1}\right) + (y - y_1)(y + y_1) = 0,$$

i.e. $$x^2 + y^2 - x\left(x_1 + \frac{y_1^2 + c^2}{x_1}\right) + c^2 = 0,$$

which is a circle of the λ system.

CONIC SECTIONS

26. Conic sections derive their name from the fact that they can all be obtained by plane sections of a double right-circular cone, and include (i) a pair of straight lines, (ii) a circle, (iii) a parabola, (iv) an ellipse, (v) a hyperbola.

A pair of straight lines is obtained from the double cone by a section through the common vertex cutting the double cone in two generators.

A circle is obtained by a plane section perpendicular to the common axis of the double cone.

A parabola is obtained by a plane section parallel to a generator of the double cone.

An ellipse is obtained by a section cutting only one half of the double cone, and not perpendicular to the axis or parallel to a generator.

A hyperbola is obtained by a section cutting both halves of the double cone and not passing through the common vertex.

The mathematical definition of a conic section is as follows:

A *conic section* is the locus of a point in a plane that moves so that its distance from a fixed point (*focus*) in the plane bears a constant ratio (*eccentricity*) to its distance from a fixed straight line (*directrix*) in the plane.

The magnitude of the eccentricity, which is usually denoted by e, determines the type of curve obtained, thus:

(i) for a pair of straight lines, eccentricity $e = \infty$;

(ii) for a circle, $e = 0$;

(iii) for a parabola, $e = 1$;

(iv) for an ellipse, $e < 1$;

(v) for a hyperbola, $e > 1$.

The circle is a degenerate form of ellipse and a pair of straight lines is a degenerate form of hyperbola.

Definition.—The chord through the focus of a conic parallel to the directrix is known as the *latus rectum.*

The Parabola

27. Definition.—A parabola is a conic section whose eccentricity is unity.

More fully, this would be stated as:

A *parabola* is the locus of a point that moves in a plane so that its distance from a fixed point in the plane is equal to its distance from a fixed straight line in the plane.

28. Theorem.—To find the standard form of the equation of a parabola.

N.B.—The notation used throughout is standard notation.

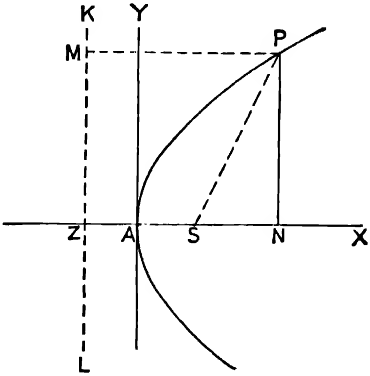
Let S (the fixed point) be the focus, and KL (fixed straight line) be the directrix. Draw SZ perpendicular to KL and choose ZS as the axis of x. Let A, the mid-point of SZ, be the origin of co-ordinates, and SZ be of length $2a$. Hence A lies on the parabola. AY perpendicular to SZ is the y-axis, and $P \equiv (x, y)$ is any point on the parabola with this choice of origin and axes. PM is the perpendicular from P on KL, and PN is the ordinate of P.

Now the focus $S \equiv (a, 0)$.

From the diagram

$$PM = NZ = AZ + AN = a + x.$$

Also
$$PS^2 = (x - a)^2 + y^2.$$



From the definition of the parabola

$$PS^2 = PM^2,$$

i.e.
$$(x - a)^2 + y^2 = (a + x)^2,$$

i.e.
$$y^2 = 4ax,$$

which is the required *canonical* (simplest) form of the equation of the parabola, since P is any point on the locus.

29. Simple properties of the parabola $y^2 = 4ax$.

Since y^2 is always positive, it follows that x is always positive if a is positive, and negative if a is negative. Therefore, when a is positive, the curve lies entirely to the right of OY, and when a is negative, it lies entirely to the left of OY.

As $x \to \infty$ so $y \to \pm \infty$, and therefore both portions of the curve extend to infinity.

For any value of x there are two equal and opposite values of y, therefore the curve is symmetrical about OX, which is known as the *axis* of the curve.

The origin A is known as the *vertex* of the parabola.

30. Theorem.—To prove that the latus rectum of the parabola $y^2 = 4ax$ is of length $4a$.

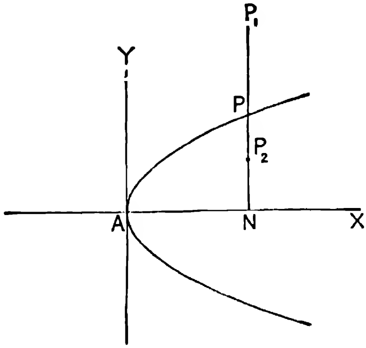
When $x = a$, i.e. at the focus S,

$$y^2 = 4a^2, \qquad \therefore\ y = \pm 2a.$$

Hence the length of the latus rectum $= 2a + 2a = 4a$, and the length of the semi-latus rectum $= 2a$.

31. Theorem.—To prove that, for all points inside the parabola $y^2 = 4ax$, $y^2 - 4ax < 0$; for all points on the curve $y^2 = 4ax$; and for all points outside the curve $y^2 - 4ax > 0$.

Let $P \equiv (x, y)$ be any point on the curve $y^2 = 4ax$, and P_1, P_2 be points outside and inside the curve respectively, such that P_1PP_2 is a straight line parallel to OY.



Let $P_1 \equiv (x, y_1)$ and $P_2 \equiv (x, y_2)$. Since P lies on the parabola, its co-ordinates satisfy the equation of the curve.

Thus in this case $\qquad y^2 - 4ax = 0.$

If N be the point in which P_1P_2 meets AX,

$$P_1N > PN, \text{ i.e. } P_1N^2 - PN^2 > 0,$$

i.e. $$y_1^2 - y^2 > 0, \text{ or } y_1^2 - 4ax > 0.$$

Similarly, $P_2N < PN$, i.e. $\quad y_2 < y$,

i.e. $$y_2^2 - y^2 < 0, \text{ or } y_2^2 - 4ax < 0.$$

32. Theorem.—To find the condition that the straight line $y = mx + c$ shall touch the parabola $y^2 = 4ax$.

$$y = mx + c. \quad \ldots\ldots\ldots \quad (38)$$

$$y^2 = 4ax. \quad \ldots\ldots\ldots \quad (39)$$

Substituting from (38) in (39) for the points of intersection of the line and parabola,

$$(mx + c)^2 = 4ax,$$

i.e. $$m^2x^2 + 2x(mc - 2a) + c^2 = 0. \quad . \quad . \quad . \quad . \quad (40)$$

If the line (38) is to touch the parabola (39), it is necessary that the equation (40) shall have coincident roots, and the condition for this is

$$(mc - 2a)^2 = m^2c^2,$$

i.e. $$4a^2 - 4amc = 0,$$

i.e. $$c = a/m \qquad \text{(since } a \neq 0\text{).}$$

Thus the line $$\boldsymbol{y = mx + a/m}$$

is a tangent to the parabola $y^2 = 4ax$ for all values of m, and this is known as the *slope equation* of the tangent.

33. Theorem.—To find the equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1), using the secant method.

[*N.B.*—The result is most readily obtained by using calculus.]

Let P $\equiv (x_1, y_1)$, Q $\equiv (x_2, y_2)$ be two points on the parabola

$$y^2 = 4ax. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Then $$y_1{}^2 = 4ax_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

$$y_2{}^2 = 4ax_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

(42) — (43) gives $$y_1{}^2 - y_2{}^2 = 4a(x_1 - x_2). \quad . \quad . \quad . \quad . \quad . \quad (44)$$

The equation of PQ is $$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

From (44) × (45), $(y - y_1)(y_1 + y_2) = 4a(x - x_1)$ is the equation of PQ.

Replacing y_2 by y_1, the equation of the tangent at (x_1, y_1) is

$$2y_1(y - y_1) = 4a(x - x_1),$$

i.e. $$yy_1 = 2ax - 2ax_1 + y_1{}^2,$$

i.e. $$yy_1 = 2ax - 2ax_1 + 4ax_1 \qquad \text{[using (42)],}$$

i.e. $$\boldsymbol{yy_1 = 2a(x + x_1).}$$

When $x_1 = y_1 = 0$, the equation is that of the tangent at the vertex A, and is $x = 0$, i.e. the axis of y.

34. Theorem.—To find the equation of the normal at the point (x_1, y_1) to the parabola $y^2 = 4ax$.

The equation of the tangent to the parabola at (x_1, y_1) is $yy_1 = 2a(x + x_1)$, and its slope is $2a/y_1$.

Therefore the slope of the normal is $-y_1/2a$, and the equation of the normal at (x_1, y_1) is

$$(y - y_1) = -\frac{y_1}{2a}(x - x_1). \quad . \quad . \quad . \quad . \quad (46)$$

From this result can be deduced the slope form of the equation of the normal as follows:

Let m be the slope of the normal, then

$$m = -\frac{y_1}{2a}, \text{ i.e. } y_1 = -2am.$$

Also, $$x_1 = y_1{}^2/4a = 4a^2m^2/4a = am^2.$$

Therefore the slope equation of normal is, from (46),

$$y + 2am = m(x - am^2),$$

i.e. $$y = mx - 2am - am^3.$$

This is a very useful equation, and shows that three normals can be drawn from a point (h, k) to the parabola $y^2 = 4ax$, that have their slopes m satisfying the cubic equation

$$am^3 + m(2a - h) + k = 0.$$

35. Theorem.—To prove that the locus of the middle points of a system of parallel chords of a parabola is a straight line parallel to the axis of the parabola.

Let the equation of the parabola be

$$y^2 = 4ax. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Let $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$ be the extremities of one of the chords of the parallel system whose slopes are m.

Since P and Q lie on the parabola (47),

$$y_1{}^2 = 4ax_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

$$y_2{}^2 = 4ax_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

From (48) — (49),

$$(y_1 - y_2)(y_1 + y_2) = 4a(x_1 - x_2). \quad . \quad . \quad . \quad (50)$$

The equation of PQ is

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

From (50) $\times$ (51), the equation of the chord PQ is

$$(y - y_1)(y_1 + y_2) = 4a(x - x_1).$$

The slope of PQ is therefore

$$\frac{4a}{y_1 + y_2} = m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

If $\bar{y}$ be the ordinate of the mid-point of PQ, then

$$2\bar{y} = y_1 + y_2,$$

and equation (52) becomes $\quad m = 2a/\bar{y}.$

Since m is a constant, the equation of the locus of the mid-point of PQ is

$$y = 2a/m,$$

which is a *straight line parallel to the axis of the parabola.*

From equation (52), it can also be seen that the sum of the ordinates of the extremities of a chord of a parabola belonging to a parallel system of chords is a constant.

36. Definition.—The locus of the mid-points of parallel chords of any conic is known as a *diameter*, and the chords it bisects are known as *ordinates* of that diameter. The points in which a diameter cuts a conic are called the *extremities* of the diameter, and in the case of a parabola there is only one extremity.

37. Theorem.—To prove that the tangent at an extremity of a diameter is parallel to the chords bisected by that diameter.

Considering a chord of the parallel system, let the extremities of the chord move closer and closer together, the chord in question being still bisected by the particular diameter, and hence the chord will still be a member of the parallel system of chords.

In the limiting case, the chord becomes the tangent at the extremity of the diameter and must still be parallel to the chords of the system.

38. Theorem.—To prove that the tangents at the ends of a chord intersect on the diameter bisecting the chord.

Let the equation to the parabola be

$$y^2 = 4ax, \quad \ldots\ldots\ldots \quad (53)$$

and $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$ be the ends of the chord in question.

The equations of the tangents at P and Q are respectively

$$yy_1 = 2a(x + x_1), \quad \ldots\ldots\ldots \quad (54)$$

and

$$yy_2 = 2a(x + x_2). \quad \ldots\ldots\ldots \quad (55)$$

From (54) — (55), for the point of intersection of the tangents,

$$y(y_1 - y_2) = 2a(x_1 - x_2),$$

i.e.

$$y = \frac{2a(x_1 - x_2)}{y_1 - y_2}. \quad \ldots\ldots\ldots \quad (56)$$

Since P and Q lie on the parabola

$$y_1^2 = 4ax_1,$$

$$y_2^2 = 4ax_2,$$

$$\therefore\ y_1^2 - y_2^2 = 4a(x_1 - x_2),$$

i.e.

$$\frac{y_1 + y_2}{2} = \frac{2a(x_1 - x_2)}{y_1 - y_2}.$$

Using this in (56), for the point of intersection of the tangents,

$$y = (y_1 + y_2)/2.$$

But it has been shown that the diameter through the mid-point of the chord PQ is parallel to OX, and since the mid-point of PQ has ordinate $(y_1 + y_2)/2$, it follows that the *diameter bisecting* PQ *passes through the point of intersection of the tangents at* P *and* Q.

39. Parametric representation of a parabola.

In the case of the parabola $y^2 = 4ax$, it can readily be verified that the point $(at^2, 2at)$ lies on the parabola for all values of t.

Thus the parametric equations of the parabola are

$$x = at^2,\ y = 2at.$$

40. Theorem.—To prove that, if the parameter of one end of a focal chord of a parabola be t_1, then the parameter of the other extremity will be $-1/t_1$, where the equations of the parabola are

$$x = at^2,\ y = 2at.$$

Let t_2 be the parameter of Q, the other extremity of the focal chord. Then, if P be the point whose parameter is t_1, the equation of PQ is

$$\frac{y - 2at_1}{2at_1 - 2at_2} = \frac{x - at_1^2}{at_1^2 - at_2^2},$$

i.e. $$(y - 2at_1)(t_1 + t_2) = 2(x - at_1^2).$$

Since the focus $(a, 0)$ lies on PQ,

$$-2at_1(t_1 + t_2) = 2(a - at_1^2),$$

i.e. $$-at_1t_2 = a,$$

i.e. $$t_2 = -1/t_1 \qquad (a \neq 0).$$

41. Theorem.—To prove that the tangents at the ends of a focal chord of a parabola intersect at right angles on the directrix.

Taking $x = at^2$, $y = 2at$ as the equations of the parabola, with P and Q the extremities of any focal chord, if the parameter of P be t_1, then the parameter of Q is $-1/t_1$,

i.e. $$\text{P} \equiv (at_1^2, 2at_1),\ \text{Q} \equiv \left(\frac{a}{t_1^2}, -2a/t_1\right).$$

Using $yy_1 = 2a(x + x_1)$ as the equation of the tangent at (x_1, y_1) to the parabola, the equations of the tangents at P and Q will respectively be

$$2at_1y = 2a(x + at_1^2), \quad \ldots \ldots \quad (57)$$

and $$-\frac{2a}{t_1}y = 2a\left(x + \frac{a}{t_1^2}\right),$$

i.e. $$-2at_1y = 2a(t_1^2x + a). \quad \ldots \ldots \quad (58)$$

Adding (57) and (58) for the point of intersection of the tangents at P and Q,

$$0 = 2a[x(1 + t_1^2) + a(1 + t_1^2)],$$

i.e. $$x + a = 0,$$

since $1 + t_1^2 \neq 0$ and $a \neq 0$.

This is the equation of the directrix, and hence the tangents at the ends of a focal chord meet on the directrix.

From (57), the slope of the tangent at P is $1/t_1$.

From (58), the slope of the tangent at Q is $-t_1$.

Therefore the product of the slopes of the tangents at P and Q is -1, and the tangents are at right angles.

Thus the tangents at the ends of a focal chord intersect at right angles on the directrix.

Note.—This result is proved geometrically on p. 279.

42. Theorem.—If the normal at $P \equiv (x_1, y_1)$, on the parabola $y^2 = 4ax$, cuts OX in G, and PN is the ordinate of P, to prove that the length of the subnormal is $2a$.

The equation of the normal at P is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1).$$

At the point G, where $y = 0$,

$$-y_1 = -\frac{y_1}{2a}(x - x_1),$$

i.e.
$$2a = (x - x_1) = \text{AG} - \text{AN} \quad \text{(A is the vertex)}$$
$$= \text{NG}.$$
$$\therefore\ \text{NG} = 2a.$$

43. Theorem.—Two tangents can be drawn to a parabola from any point. They will be real, coincident, or imaginary, according as the point lies outside, on, or within the curve.

The equation of the tangent of slope m to the parabola $y^2 = 4ax$ is

$$y = mx + a/m. \qquad \ldots \ldots \quad (59)$$

If the line (59) passes through the point (h, k),

$$k = mh + a/m,$$

i.e.
$$m^2h - mk + a = 0. \qquad \ldots \ldots \quad (60)$$

The equation (60) is a quadratic in m, and hence two tangents can be drawn from the point (h, k) to the parabola $y^2 = 4ax$.

These tangents will be real, coincident, or imaginary, according as the roots of equation (60) are real, coincident, or imaginary,

i.e. according as $\quad k^2 >$ or $=$ or $< 4ah$.

But these are the conditions that the point (h, k) shall lie outside, on, or within the curve $y^2 = 4ax$. Hence the tangents from the point (h, k) to the curve $y^2 = 4ax$ will be real, coincident, or imaginary, according as the point (h, k) lies outside, on, or within the curve.

44. Theorem.—To find the polar of the point (h, k) with respect to the parabola $y^2 = 4ax$ (i.e. the line joining the real, coincident, or imaginary points of contact of the tangents drawn from (h, k) to the parabola).

Let $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$ be the points of contact of the two tangents from the point (h, k) to the given parabola.

The equations of the tangents at P and Q are respectively

$$yy_1 = 2a(x + x_1), \quad \ldots \ldots \quad (61)$$

and

$$yy_2 = 2a(x + x_2). \quad \ldots \ldots \quad (62)$$

Since the point (h, k) lies on the lines (61) and (62),

$$ky_1 = 2a(h + x_1), \quad \ldots \ldots \quad (63)$$

$$ky_2 = 2a(h + x_2). \quad \ldots \ldots \quad (64)$$

But (63) and (64) are the conditions that P and Q shall lie on the straight line $ky = 2a(x + h)$, which is therefore the required equation.

45. Theorem.—To prove that, if the polar of P with respect to the parabola $y^2 = 4ax$ passes through Q, then the polar of Q with respect to the parabola will pass through P.

Let $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$.

Then the polars of P and Q with respect to the parabola will be

$$yy_1 = 2a(x + x_1), \quad \ldots \ldots \quad (65)$$

and

$$yy_2 = 2a(x + x_2). \quad \ldots \ldots \quad (66)$$

Since the line (65) passes through Q,

$$y_2y_1 = 2a(x_2 + x_1).$$

But this is the condition that (x_1, y_1) shall lie on the line (66).

Hence if the polar of P *passes through* Q, *the polar of* Q *will pass through* P.

46. Theorem.—If the polars of P and Q with respect to the parabola $y^2 = 4ax$ pass through R, then the polar of R is the line PQ.

Since the polar of P passes through R, the polar of R passes through P. Since the polar of Q passes through R, the polar of R passes through Q. Hence the polar of R passes through both P and Q and therefore is the line PQ.

47. Theorem.—To prove that the polar of the focus of a parabola with respect to the parabola is the directrix.

Let the equation of the parabola be $y^2 = 4ax$. Then $S \equiv (a, 0)$, and the polar of S is the line

$$0 = 2a(x + a),$$

i.e. $$x + a = 0 \qquad \text{(since } a \neq 0\text{).}$$

But this is the equation of the directrix, and therefore the *polar of the focus is the directrix.*

If Q be any point on the directrix, since the polar of S passes through Q, the polar of Q passes through S, i.e. *the focus* S *lies on the chord of contact of the tangents drawn from any point on the directrix to the parabola.*

Conversely, the tangents at the ends of a focal chord will meet on the directrix.

Example 6 (L.U.).—If the normals at the points $P \equiv [t_1]$, $Q \equiv [t_2]$, on the parabola $y^2 = 4ax$, meet on the curve at the point $R \equiv [T]$, prove that t_1 and t_2 are the roots of the equation $t^2 + tT + 2 = 0$.

Show that, for all positions of R on the curve, the chord PQ passes through a fixed point, and find the co-ordinates of the pole of PQ for a given position of R.

The given parabola is $$y^2 = 4ax. \qquad \text{(i)}$$

The equation of the normal at (x_1, y_1) to the parabola (i) is

$$(y - y_1) + \frac{y_1}{2a}(x - x_1) = 0.$$

Therefore the equations of the normals at $P \equiv (at_1^2, 2at_1)$ and $Q \equiv (at_2^2, 2at_2)$ are respectively

$$(y - 2at_1) + t_1(x - at_1^2) = 0, \qquad \text{(ii)}$$

and $$(y - 2at_2) + t_2(x - at_2^2) = 0. \qquad \text{(iii)}$$

Since $R \equiv (aT^2, 2aT)$ lies on the line (ii),

$$(2aT - 2at_1) + t_1(aT^2 - at_1^2) = 0,$$

i.e. $$2a(T - t_1) + at_1(T - t_1)(T + t_1) = 0,$$

i.e. $$2 + t_1(T + t_1) = 0 \qquad \text{(since } T \neq t_1 \text{ and } a \neq 0\text{),}$$

i.e. $$t_1^2 + t_1T + 2 = 0. \qquad \text{(iv)}$$

Similarly, since R lies on the line (iii),

$$t_2^2 + t_2T + 2 = 0. \qquad \text{(v)}$$

From (iv) and (v) it can be seen that t_1 and t_2 are the roots of the equation

$$t^2 + tT + 2 = 0.$$

From this equation
$$t_1 + t_2 = -T, \quad \ldots \ldots \quad \text{(vi)}$$
$$t_1t_2 = 2. \quad \ldots \ldots \quad \text{(vii)}$$

The equation of the chord PQ is

$$\frac{y - 2at_1}{2at_1 - 2at_2} = \frac{x - at_1^2}{at_1^2 - at_2^2},$$

i.e.
$$(y - 2at_1)(t_1 + t_2) = 2(x - at_1^2),$$

i.e.
$$y(t_1 + t_2) = 2x + 2at_1t_2,$$

i.e. [using (vi) and (vii)]
$$-yT = 2x + 4a. \quad \ldots \ldots \quad \text{(viii)}$$

This result shows that the line PQ passes through the fixed point $(-2a, 0)$.

If (h, k) be the pole of (viii) with respect to the parabola (i), then the equation

$$ky = 2a(x + h)$$

must be equivalent to the equation (viii), since they represent the same line.

$$\therefore \quad \frac{k}{-T} = \frac{2a}{2} = \frac{2ah}{4a} \quad \text{(comparing coefficients),}$$

i.e.
$$k = -aT, \; h = 2a.$$

Thus the pole of PQ is $(2a, -aT)$.

Example 7 (L.U.).—Find the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

P is the point $(aT^2, 2aT)$ on this parabola; show that, if $T^2 > 8$, there are two real normals to the curve which pass through P in addition to the normal at P.

Q and R are the points at which the normals will pass through P; M is the mid-point of QR and N is the mid-point of PM. Show that, as P moves along the parabola, N describes a parabola, and QR passes through a fixed point.

$$y^2 = 4ax. \quad \ldots \ldots \quad \text{(i)}$$

Differentiating (i) with respect to x,

$$2y\frac{dy}{dx} = 4a, \quad \therefore \frac{dy}{dx} = \frac{2a}{y}.$$

Therefore the slope of the tangent at the point $(at^2, 2at)$ is $2a/2at = 1/t$, and the slope of the normal will be $(-t)$.

Hence the equation of the normal at $(at^2, 2at)$ is

$$y - 2at = -t(x - at^2),$$

$$tx + y = 2at + at^3.$$

If $P \equiv (aT^2, 2aT)$ lie on the above normal,

$$aT^2t + 2aT = 2at + at^3,$$

i.e.
$$aT^2t - at^3 + 2aT - 2at = 0,$$

i.e.
$$t(T^2 - t^2) + 2(T - t) = 0 \qquad (a \neq 0),$$

i.e.
$$(T - t)(Tt + t^2 + 2) = 0,$$

$$\therefore\ t = T, \text{ or } t = [(-T \pm \sqrt{(T^2 - 8)}]/2. \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Thus there are two real normals that pass through P in addition to the normal at P if $T^2 > 8$.

From the equation (i) using $t_1 = [(-T + \sqrt{(T^2 - 8)}]/2$,

$$t_2 = [(-T - \sqrt{(T^2 - 8)}]/2,$$

$$Q \equiv (at_1^2, 2at_1),\ R \equiv (at_2^2, 2at_2),$$

$$\therefore\ M \equiv \{a(t_1^2 + t_2^2)/2,\ a(t_1 + t_2)\}.$$

But
$$t_1 + t_2 = -T,\quad \frac{t_1^2 + t_2^2}{2} = \frac{2(T^2 + T^2 - 8)}{4 \times 2} = \frac{T^2 - 4}{2}.$$

$$\therefore\ M \equiv \{\tfrac{1}{2}a(T^2 - 4),\ -aT\}.$$

N being the mid-point of PM and taken as $(\bar{x}, \bar{y})$,

$$\bar{x} = \{aT^2 + \tfrac{1}{2}a(T^2 - 4)\}/2 = \tfrac{1}{4}a(3T^2 - 4), \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

$$\bar{y} = (2aT - aT)/2 = aT/2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Substituting for T from (iii) in (ii),

$$\bar{x} = \tfrac{1}{4}a(3 \times 4\bar{y}^2/a^2 - 4) = \frac{1}{a}(3\bar{y}^2 - 4a^2),$$

i.e.
$$3\bar{y}^2 = a(\bar{x} + 4a),$$

which is the equation of a parabola, using $(\bar{x}, \bar{y})$ as the variable point, therefore N describes a parabola.

The equation of QR is

$$\frac{x - at_1^2}{at_1^2 - at_2^2} = \frac{y - 2at_1}{2at_1 - 2at_2},$$

i.e.
$$2(x - at_1^2) = (t_1 + t_2)(y - 2at_1),$$

i.e.
$$2x - 2at_1^2 = -Ty - 2at_1^2 - 2at_1t_2,$$

i.e.
$$2x = -Ty - 4a$$

[using $t_1 + t_2 = -T$, $t_1t_2 = +2$, as in Example 6, equations (vi) and (vii)].

This is satisfied by the point $(-2a, 0)$, therefore QR passes through the fixed point $(-2a, 0)$.

Example 8 (L.U.).—Find the equation of the polar of the point (x_1, y_1) with respect to the parabola $y^2 = 4ax$.

Prove that the locus of the points of intersection of the polars of points on the straight line $y = c$, with respect to the parabolas $y^2 = 4ax$, $x^2 = 4ay$, is the hyperbola $2ax^2 - cxy + 4a^2(y + c) = 0$.

The equation of the polar of (x_1, y_1), with respect to the parabola $y^2 = 4ax$, has been shown to be

$$yy_1 = 2a(x + x_1).$$

Any point on the line $y = c$ can be taken as (x_1, c), and its polars, with respect to the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, are respectively

$$yc = 2a(x + x_1), \quad \ldots \quad \text{(i)}$$

$$xx_1 = 2a(y + c). \quad \ldots \quad \text{(ii)}$$

The required locus will be obtained by eliminating the variable x_1 between equations (i) and (ii).

From (i), $\qquad x_1 = yc/2a - x.$

Using this in (ii), $\qquad xyc/2a - x^2 = 2a(y + c),$

i.e. $\qquad cxy - 2ax^2 = 4a^2(y + c),$

i.e. $\qquad 2ax^2 - cxy + 4a^2(y + c) = 0.$

Example 9 (L.U.).—Find the co-ordinates of the point of intersection of the tangents at the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, on the parabola $y^2 = 4ax$. If two perpendicular tangents to the parabola meet at T and the corresponding normals meet at N, show that, for all such pairs of tangents, TN is parallel to the axis of the parabola, and the locus of N is another parabola.

The equation of the tangent at (x_1, y_1) to the parabola $y^2 = 4ax$ is

$$yy_1 = 2a(x + x_1).$$

Therefore the equations of the tangents at $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ will be

$$2at_1y = 2a(x + at_1^2),$$

i.e. $\qquad t_1y = x + at_1^2, \quad \ldots \quad \text{(i)}$

and $\qquad t_2y = x + at_2^2. \quad \ldots \quad \text{(ii)}$

For the point of intersection of the tangents, taking (i)—(ii),

$$y(t_1 - t_2) = at_1^2 - at_2^2.$$

$$\therefore\ y = a(t_1 + t_2) \qquad (t_1 \neq t_2).$$

From (i), $\qquad at_1(t_1 + t_2) = x + at_1^2, \quad \therefore\ x = at_1t_2.$

Therefore the point of intersection T of the tangents is

$$\{at_1t_2,\ a(t_1 + t_2)\}. \quad \ldots \quad \text{(iii)}$$

The slope of the normal at $(at_1^2, 2at_1)$ will be $(-t_1)$, therefore the equation of the normal at this point will be

$$(y - 2at_1) = -t_1(x - at_1^2),$$

i.e. $\qquad t_1x + y = 2at_1 + at_1^3. \quad \ldots \quad \text{(iv)}$

Similarly, the equation of the normal at $(at_2^2, 2at_2)$ will be

$$t_2x + y = 2at_2 + at_2^3. \quad \ldots \quad \text{(v)}$$

For the point of intersection of these normals, taking (iv) — (v),

$$x(t_1 - t_2) = 2a(t_1 - t_2) + a(t_1 - t_2)(t_1^2 + t_1t_2 + t_2^2),$$

$$\therefore\ x = 2a + a(t_1^2 + t_1t_2 + t_2^2) \qquad (t_1 \neq t_2),$$

and using this in (iv),

$$y = 2at_1 + at_1^3 - 2at_1 - at_1(t_1^2 + t_1t_2 + t_2^2)$$

$$= -at_1t_2(t_1 + t_2).$$

Therefore the point of intersection N of the normals is given by

$$x = 2a + a(t_1^2 + t_1t_2 + t_2^2),\ y = -at_1t_2(t_1 + t_2). \quad . \quad . \quad \text{(vi)}$$

If the tangents at $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ are perpendicular, then the product of the slopes ($1/t_1$ and $1/t_2$) is equal to -1,

i.e. $$1/(t_1t_2) = -1,$$

i.e. $$t_1t_2 = -1.$$

Using this in (iii) and (vi),

$$T \equiv \{-a,\ a(t_1 + t_2)\};$$

$$N \equiv \{2a + a(t_1^2 - 1 + t_2^2),\ a(t_1 + t_2)\}.$$

Since T and N have the same ordinate, TN is parallel to OX, i.e. TN is parallel to the axis of the parabola.

For N, $$x = 2a + a(t_1^2 - 1 + t_2^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vii)}$$

$$y = a(t_1 + t_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(viii)}$$

From (viii), $$y^2 = a^2(t_1^2 + 2t_1t_2 + t_2^2)$$

$$= a^2(t_1^2 - 2 + t_2^2) \qquad (\text{using } t_1t_2 = -1).$$

$$\therefore\ a(t_1^2 - 1 + t_2^2) = y^2/a + a.$$

Using this in (vii), the locus of N is given by

$$x = 2a + y^2/a + a,$$

i.e. $$y^2 = a(x - 3a),$$

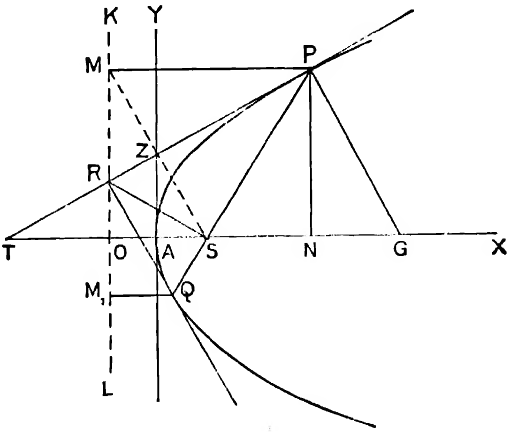
i.e. the locus of N is a parabola, vertex $(3a, 0)$, axis OX.

Geometrical Properties of the Parabola

48. In the diagram the following standard notation is used for the parabola:

A is the vertex, AX the axis of x, and AY the axis of y. $P \equiv (x_1, y_1)$ is any point on the parabola $y^2 = 4ax$, and PN its ordinate. PM is the perpendicular from P on the directrix KL. S is the focus and SM meets the y-axis in Z. PT is the tangent at P to the parabola meeting the directrix at R and the x-axis at T. O is the point in which the x-axis meets the directrix, and PG is the normal at P meeting the x-axis

at G. RQ is the other tangent from R to the parabola touching it at Q, and QM_1 is the perpendicular from Q on KL.



49. Theorem.—To prove that TA = AN, and TS = SP.

The equation of the tangent at P is

$$yy_1 = 2a(x + x_1).$$

Where this tangent cuts the x-axis, i.e. at T, $y = 0$, therefore for T

$$x + x_1 = 0, \text{ i.e. } x = -x_1,$$

i.e. $\quad AT = AN.$

From this result, since $\quad AS = OA = a,$

$$AT + AS = AN + OA.$$

$$\therefore\ TS = ON.$$

From the diagram, $\quad ON = PM.$

But, by definition, $\quad PM = SP,$

$$\therefore\ TS = PS.$$

50. Theorem.—To prove that the tangent PT bisects the angle SPM.

Since $\quad ST = SP \quad$ (proved),

$$\angle STP = \angle SPT.$$

But PM is parallel to TS, $\therefore\ \angle STP = \angle MPT \quad$ (alt. $\angle$s).

$$\therefore\ \angle SPT = \angle MPT,$$

i.e. the tangent at P bisects $\angle SPM$.

51. Theorem.—To prove that the angle subtended by the length of the tangent from the point of contact to the directrix of the parabola subtends a right angle at the focus (i.e. $\angle RSP = 90°$).

Comparing triangles RSP and MPR,

$$PR \text{ is common;}$$

$$PS = PM \qquad \text{(definition);}$$

$$\angle SPR = \angle MPR \qquad \text{(proved).}$$

$$\therefore \triangle SPR \equiv \triangle MRP \qquad \text{(2 sides + incl. } \angle\text{).}$$

$$\therefore \angle RSP = \angle PMR = 90°.$$

52. Theorem.—To prove that SM is perpendicular to PR.

Since $SP = PM$, and RP is the bisector of the vertical angle SPM of $\triangle SPM$, it follows that PR is perpendicular to SM and bisects SM.

53. Theorem.—To prove that the line through the focus S, perpendicular to the tangent PT, meets this tangent on the tangent at the vertex A.

It has been proved that the line from S perpendicular to PT is the line SM, and SM is bisected by PT.

Now A is the mid-point of OS and AZ is parallel to OM, therefore Z is the mid-point of SM.

But PT passes through the mid-point of SM, therefore Z must lie on PT, which proves the theorem.

54. Theorem.—To prove that the tangents at the ends of a focal chord (i.e. a chord of the parabola passing through the focus) meet at right angles on the directrix.

Since P and Q are the points of contact of the tangents from R to the parabola, by Section 51,

$$\angle RSQ = 90°,$$

and

$$\angle RSP = 90°.$$

$$\therefore \angle RSP + \angle RSQ = 180° = \text{straight angle.}$$

Therefore P, S, and Q are in the same straight line, i.e. PSQ is a focal chord.

By Section 50,

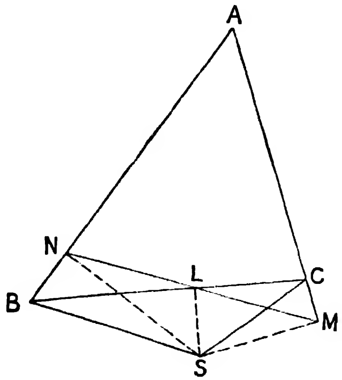
$$\angle RPS = \tfrac{1}{2}\angle SPM, \text{ and } \angle RQS = \tfrac{1}{2}\angle SQM_1,$$

$$\begin{aligned} \therefore\ \angle SPR + \angle SQR &= \tfrac{1}{2}(\angle SPM + \angle SQM_1) \\ &= \tfrac{1}{2} \times 180^\circ \quad \text{(PM and QM}_1\text{ are parallel)} \\ &= 90^\circ. \end{aligned}$$

Therefore the tangents at the ends of a focal chord intersect at right angles on the directrix.

Example 10 (L.U.).—Show that the foot of the perpendicular from the focus of a parabola to any tangent lies on the tangent at the vertex.

Three tangents to a parabola form the triangle ABC. Show that the circumcircle of triangle ABC passes through the focus S.



The first part of the question was proved in Section 53.

Let L, M, N be the feet of the perpendiculars from the focus S on the tangents BC, CA, AB respectively.

By the first part of the question, L, M, N lie on the tangent at the vertex of the parabola and are therefore collinear.

Since $\angle SNA = \angle SMA = 90^\circ$,

$$\therefore\ \angle NSM + \angle NAM = 180^\circ. \quad \ldots\ldots\ldots \quad \text{(i)}$$

Since $\angle SLB = \angle SNB = 90^\circ$,

S, L, N, B are concyclic, and

$$\therefore\ \angle NLB = \angle NSB \qquad \text{(same segment).}$$

Similarly, S, M, C, L are concyclic,

and $\angle CSM = \angle CLM$ (same segment).

But $\angle NLB = \angle CLM$ (vert. opp. $\angle$s),

$$\therefore\ \angle NSB = \angle CSM.$$

Now $\qquad \angle BSC = \angle BSN + \angle NSC$
$$= \angle CSM + \angle NSC$$
$$= \angle NSM.$$

Using this in (i), $\qquad \angle BSC + \angle NAM = 180°.$

Therefore S, C, A, B are concyclic (opp. $\angle$s supplementary), i.e. S lies on the circumcircle of $\triangle ABC$.

EXAMPLES ON CHAPTER IX

All the following examples are taken from London University examination papers.

1. Find the condition that the equation

$$F(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

shall represent a pair of straight lines and, in this case, find the product of the perpendiculars upon them from the point (p, q).

Show also that $F(x, y) = 0$ represents a pair of parallel straight lines if $h^2 = ab$, $af^2 = bg^2$ (see general conic).

2. Find the equation of the bisectors of the angles between the two straight lines $ax^2 + 2hxy + by^2 = 0$.

Find separately the equations of the bisectors between the straight lines drawn through the point $(-2, 1)$ parallel to the straight lines

$$3x^2 + 12xy - 2y^2 = 0.$$

3. Find the condition that the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents two straight lines.

Assuming that this condition be satisfied, A is the intersection of the straight lines. Parallel straight lines are drawn through the origin O and they intersect the other lines in B and C. Find the equations of the diagonals OA and BC of the parallelogram formed, and show that the parallelogram is a square if $a + b = 0$ and $h(g^2 - f^2) = fg(a - b)$.

4. Show that the equation $ax^2 + 2hxy + by^2 = 0$ represents two straight lines, and find the angle between them.

Form the equation of the straight lines joining the origin to the points given by the equations $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, $px + qy + r = 0$, and write down the condition that these lines should be at right angles.

If this condition be satisfied, show that the locus of the foot of the perpendicular from the origin to the line $px + qy + r = 0$ is

$$(a + b)(x^2 + y^2) + 2gx + 2fy + c = 0.$$

5. Show that $ax^2 + 2hxy + by^2 = 0$ represents two straight lines through the origin O, and show that $h(x^2 - y^2) = (a - b)xy$ is the equation of the bisectors of the angles between them.

The straight line $px + qy + r = 0$ cuts the straight lines

$$ax^2 + 2hxy + by^2 = 0$$

at A and B. Find the co-ordinates of C, if the figure OACB is a parallelogram, and prove that the figure OACB is a rhombus if $h(p^2 - q^2) = (a - b)pq$.

6. Prove that the equation $x^2 + y^2 + 2\lambda x + c = 0$, where λ is variable and c constant, represents a coaxal system of circles.

Find the equations of the circles of the system $x^2 + y^2 + 2\lambda x + 2 = 0$, which touch the line $x + y = 4$, and find the distance between the points of contact.

7. Show that the circles

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$
$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

cut orthogonally if $$2gg_1 + 2ff_1 = c + c_1.$$

Find the general equation of a circle which cuts orthogonally each of the two circles

$$x^2 + y^2 - 6ax + 5a^2 = 0,$$
$$x^2 + y^2 - 6ay + 5a^2 = 0.$$

Show that all such circles belong to a coaxal system with real common points, and find the co-ordinates of these points.

8. Find the equation of the circle which touches the y-axis and is such that the polar of the origin with respect to it is $x + py = 1$; also find the equation of the other tangent drawn to it from the origin.

If p varies, show that the locus of the centre of the circle is a parabola, and that the circle itself always touches the fixed circle

$$2(x^2 + y^2) = x.$$

9. (i) A variable circle passes through a fixed point and cuts a fixed circle at the ends of a variable diameter of the fixed circle. Show that the locus of the centre of the variable circle is a straight line.

(ii) The polar of the origin with respect to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

intersects the circle at P and Q. Show that the equation of the circle on PQ as diameter is

$$(f^2 + g^2)(x^2 + y^2) + 2cgx + 2cfy + 2c^2 - c(f^2 + g^2) = 0.$$

10. Find the co-ordinates of the centre and the length of the radius of the circle inscribed in the triangle whose sides lie along the straight lines

$$4x - 3y + 2a = 0,$$
$$3x - 4y + 12a = 0,$$
$$3x + 4y - 12a = 0.$$

11. Find the condition that the chord of the circle $(x - c)^2 + y^2 = a^2$, which lies along $lx + my + n = 0$, may subtend a right angle at the origin of co-ordinates.

Hence, or otherwise, prove that, if a variable chord PQ of a circle with centre C subtends a right angle at a fixed internal point O, the locus of the foot of the perpendicular from O to PQ is a circle whose centre is midway between O and C.

12. Find the length of either tangent drawn from the point (x', y') to the circle

$$ax^2 + ay^2 + 2gx + 2fy + c = 0.$$

If $t_1^2 = \alpha t_2^2 + \beta t_3^2 + \gamma$, where t_1, t_2, t_3 are respectively the lengths of the tangents drawn from any point (x', y') to the three circles

$$x^2 + y^2 = a^2, \quad x^2 + y^2 = 2ax, \quad x^2 + y^2 = 2ay,$$

prove that the values of the constants α, β, γ can be found so that (x', y') lies on the fixed straight line $2x + 4y - 3a = 0$.

13. Show that the equations of any two circles can, by suitable choice of axes, be put in the form

$$x^2 + y^2 + 2gx + c = 0 \text{ and } x^2 + y^2 + 2g_1x + c = 0.$$

Given two circles, a tangent to one of the given circles at any point P on it meets the polar of P with respect to the other in P_1. Prove that the circles on PP_1 as diameter form a coaxal system, and find the limiting points of this system.

14. Show that the circles

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and

$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

are orthogonal if

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2.$$

Assuming that these circles are orthogonal, intersecting at C and D, and that their centres are A and B respectively, show that the equation of the circle through A, B, C, D is

$$2(x^2 + y^2) + 2(g_1 + g_2)x + 2(f_1 + f_2)y + c_1 + c_2 = 0.$$

Assuming that the equation of the circle on CD as diameter is written in the form

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 + \lambda\{2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2)\} = 0,$$

show that $\lambda = -r_1^2/AB^2$, where r_1 is the radius of the first circle.

15. Find the equation of the polar of the point (h, k) with respect to the circle $x^2 + y^2 = a^2$.

A point P moves so that the length of the tangent from it to the circle is equal to its distance from the polar of the fixed point (h, k). Prove that the locus of P is a parabola and find the equation of the axis of the parabola (see general conics).

16. Find the equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Three points P, Q, R are taken on this parabola. The chord PQ is parallel to the directrix, and the tangents at P and R meet on the parabola

$$y^2 = 4a(2x + a).$$

Find the locus of the point of intersection of the tangents at Q and R.

17. Show that the equation of the chord of the parabola $y^2 = 4ax$, which is bisected at (α, β), is

$$2ax - \beta y = 2a\alpha - \beta^2.$$

Find the locus of the middle points of chords of the parabola $y^2 = 4ax$, which touch the parabola $y^2 + 4ax = 0$.

18. Find the co-ordinates of the point of intersection of the tangents to the parabola $y^2 = 4ax$ at the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$.

G is the centroid of a triangle inscribed in a parabola. K is the centroid of the triangle formed by tangents to the parabola at the vertices of this triangle. Show that GK is parallel to the axis of the parabola and that the point of trisection of GK, nearer to K, lies on the parabola.

19. Find the co-ordinates of the point of intersection of the tangents drawn to the parabola $y^2 = 4ax$ at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.

Two tangents to this parabola are inclined to one another at an angle of $\pi/4$. Show that the locus of their point of intersection is a rectangular hyperbola, and find the co-ordinates of the points lying on the circle $x^2 + y^2 + 6ax = 0$, from which tangents, inclined at this angle, can be drawn to the parabola.

20. Find the equations of the tangent and normal at the point $(at^2, 2at)$ to the parabola $y^2 = 4ax$.

Q is the variable point $(aT^2, 2aT)$ on this parabola. The normals at P and P′ pass through Q. The tangents at P and Q intersect at R and the tangents at P′ and Q at R′. Show that the locus of the mid-point of RR′ is a parabola.

21. Find the equation to the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Prove that the normal at one of the extremities of the latus rectum passes through the point $(15a, 12a)$, and find the co-ordinates of the feet of the remaining normals drawn from this point.

22. Find the equation of the normal to the parabola $y^2 = 4ax$ at the point whose co-ordinates are $(at^2, 2at)$. Find also the point of intersection of the two normals at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.

If the normals at the points P and Q of the above parabola make complementary angles with the axis, show that their point of intersection lies on the parabola $y^2 = a(x - a)$.

23. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

The polar of the point A, with respect to the parabola $y^2 = 4ax$, intersects the parabola at the points P and Q, and the normals at P and Q intersect at B. If A lie on the hyperbola $xy = c^2$, show that the locus of B is a straight line.

24. Find the equation of the normal to $y^2 = 4ax$ at the point $(at^2, 2at)$.

The normals at P $(at_1^2, 2at_1)$ and Q $(at_2^2, 2at_2)$ intersect on the parabola at R $(aT^2, 2aT)$. Show that t_1 and t_2 are roots of the equation

$$t^2 + tT + 2 = 0.$$

Show that, for all values of T, the locus of the middle point of the chord PQ is a parabola.

25. Prove that the foot of the perpendicular from the focus on any tangent to a parabola lies on the tangent at the vertex.

S is a given point, and l a given line, not passing through S. A variable line SQ meets l at Q, and a line QR is drawn making a given angle RQS with QS, always in the same sense. Prove that QR envelops a parabola whose focus is S and one of whose tangents is l. Obtain the directrix and the point of contact of l.

26. ACB is a fixed diameter of a circle, centre C; P is a variable point on this circle, and R is the mid-point of the chord through P parallel to AB. The intersection of CP and AR (produced if necessary) is X. Prove that the locus of X is a parabola, whose latus rectum is a diameter perpendicular to AB.

The common tangent of the circle and this parabola touches the circle at M. Prove that $\angle ACM = 60°$.

CHAPTER X

Conic Sections—The Ellipse and Hyperbola

1. Definition.—The ellipse is a conic section whose eccentricity is less than unity.

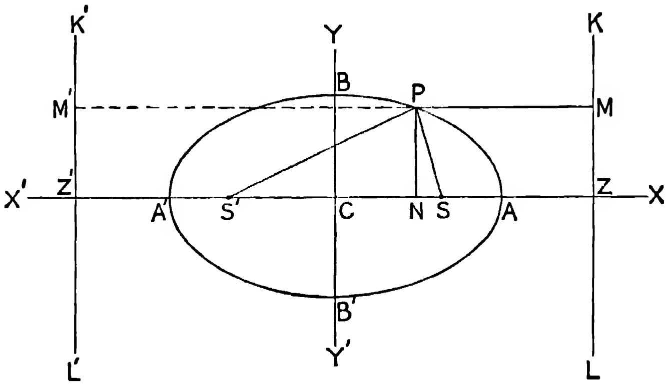
This definition is stated more fully as follows:

The ellipse is the locus of a point that moves in a plane so that its distance from a fixed point in that plane bears a constant ratio less than unity to its distance from a fixed straight line in the plane.

The notation used in the following theorem will be used throughout.

2. Theorem.—To find, with the most convenient choice of origin and axes, the simplest (canonical) form of the equation of an ellipse.

S is the focus and KL the directrix, with e (<1) the eccentricity. SZ is the perpendicular from S on KL and is produced to form the x-axis.



SZ is divided internally at A and externally at A′ in the ratio $e:1$, and the length AA′ is taken as $2a$. AA′ is bisected at C, which is taken as the origin of co-ordinates, with CY as the y-axis which cuts the ellipse in B and B′.

S′ is a point on the x-axis such that CS′ = CS, and K′L′ is a line at the same distance from CY as LK and parallel to LK, with Z′ the point in which it cuts the x-axis.

With the above choice of axes and origin, let $P \equiv (x, y)$ be any point on the ellipse with PN its ordinate and PM, PM′ the perpendiculars from P on the lines KL and K′L′ respectively.

By construction, $$\mathrm{SA} = e \,.\, \mathrm{AZ}, \qquad (1)$$

$$\mathrm{SA'} = e \,.\, \mathrm{A'Z}. \qquad (2)$$

From (1) and (2),

$$\mathrm{SA} + \mathrm{SA'} = e(\mathrm{AZ} + \mathrm{A'Z}),$$

i.e. $$\mathrm{AA'} = e[(\mathrm{CZ} - \mathrm{CA}) + (\mathrm{CZ} + \mathrm{A'C})],$$

i.e. $$2a = e[2\mathrm{CZ} - a + a] = 2e \,.\, \mathrm{CZ},$$

$$\therefore\ \mathrm{CZ} = a/e.$$

From (2) − (1), $$\mathrm{SA'} - \mathrm{SA} = e(\mathrm{A'Z} - \mathrm{AZ}),$$

i.e. $$(\mathrm{CS} + \mathrm{CA'}) - (\mathrm{CA} - \mathrm{CS}) = e \,.\, \mathrm{AA'},$$

i.e. $$2\mathrm{CS} + a - a = 2ae,$$

$$\therefore\ \mathrm{CS} = ae.$$

Hence $$\mathrm{Z} \equiv (a/e, 0), \qquad \mathrm{S} \equiv (ae, 0).$$

Now $$\mathrm{SP}^2 = (x - ae)^2 + y^2.$$

From the diagram, $$\mathrm{PM} = \mathrm{NZ} = \mathrm{CZ} - \mathrm{CN}$$
$$= a/e - x.$$

From the definition of the ellipse,

$$\mathrm{SP} = e \,.\, \mathrm{PM}, \qquad \therefore\ \mathrm{SP}^2 = e^2 \,.\, \mathrm{PM}^2,$$

i.e. $$(x - ae)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2,$$

i.e. $$x^2 - 2aex + a^2e^2 + y^2 = a^2 - 2aex + e^2x^2.$$

$$\therefore\ x^2(1 - e^2) + y^2 = a^2(1 - e^2),$$

i.e. $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1,$$

where $(1 - e^2)$ is positive, since $e < 1$.

This is the equation to the ellipse since P is any point on the locus. It is usual in the standard form of the equation to replace $a^2(1 - e^2)$ by b^2. Thus the required standard form is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $$b^2 = a^2(1 - e^2).$$

3. Theorem.—To find the length of the semi-latus rectum of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Since the latus rectum is the chord through the focus parallel to the directrix, it will be given when $x = ae$ in

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

since focus $S \equiv (ae, 0)$; i.e. for the latus rectum,

$$\frac{y^2}{b^2} = 1 - e^2,$$

i.e. $$y^2 = b^4/a^2, \qquad \text{since } b^2 = a^2(1 - e^2),$$

$$\therefore\ y = \pm b^2/a.$$

Thus the length of the semi-latus rectum is b^2/a.

4. Theorem.—To prove that the ellipse $x^2/a^2 + y^2/b^2 = 1$ is a closed curve.

The equation of the ellipse can be written

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}.$$

If $x^2 > a^2$ it is seen that y is imaginary. Similarly, if $y^2 > b^2$ it is seen that x is imaginary. Therefore there is no portion of the curve outside the limits $x = \pm a$, and $y = \pm b$, i.e. *the curve is closed.*

Note.—For every value of x such that $|x| < a$, there are two equal and opposite values of y, and for every value of y such that $|y| < b$, there are two equal and opposite values of x.

Hence the curve is symmetrical about the two axes, and from this symmetry it can be seen that S′ is a *second focus*, with K′L′ as its corresponding directrix.

5. Theorem.—To prove that, if the point (x_1, y_1) lie on the ellipse $x^2/a^2 + y^2/b^2 = 1$, then also will the point $(-x_1, -y_1)$ lie on this

ellipse, and their join will pass through the origin C, which will be the mid-point of their join.

The condition that $(-x_1, -y_1)$ shall lie on the ellipse is

$$\frac{(-x_1)^2}{a^2} + \frac{(-y_1)^2}{b^2} = 1,$$

i.e.

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1,$$

which is the condition that (x_1, y_1) lie on the ellipse. Hence, *if* (x_1, y_1) *lie on the ellipse, so also will the point* $(-x_1, -y_1)$.

The slope of the line joining C to (x_1, y_1) is y_1/x_1, and the slope of the line joining C to $(-x_1, -y_1)$ is y_1/x_1, therefore (x_1, y_1), C, *and* $(-x_1, -y_1)$ *lie on a straight line.*

The mid-point of the join of (x_1, y_1), $(-x_1, -y_1)$ is $(0, 0)$, i.e. C. Therefore C bisects the join of (x_1, y_1), $(-x_1, -y_1)$, i.e. C *bisects all chords passing through it.* For this reason C is known as the centre of the curve.

Note.—When $x = 0$, $y = \pm b$, $\therefore$ CB $=$ CB$'$ $= b$.

The line AA$'$ is known as the major axis and BB$'$ as the minor axis of the ellipse.

6. Theorem.—To prove, with standard notation, that SP $+$ S$'$P $= 2a$, for the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Using the previous diagram,

$$\begin{aligned} \text{PM}' = \text{NZ}' &= \text{CZ}' + \text{CN} \\ &= a/e + x. \end{aligned}$$

It was also shown that $\quad \text{PM} = a/e - x.$

$$\begin{aligned} \therefore\ \text{SP} &= e\,.\,\text{PM} = a - ex, \\ \text{S}'\text{P} &= e\,.\,\text{PM}' = a + ex. \end{aligned}$$

Hence

$$\begin{aligned} \textbf{SP} + \textbf{S}'\textbf{P} &= (a - ex) + (a + ex) \\ &= 2a. \end{aligned}$$

7. Theorem.—The point (x, y) lies outside, on, or inside the ellipse $x^2/a^2 + y^2/b^2 = 1$, according as $x^2/a^2 + y^2/b^2$ is greater than, equal to, or less than 1.

This result is proved in the same way as the similar results in the case of the circle and parabola.

8. Theorem.—To find the points in which the line $y = mx + c$ cuts the ellipse $x^2/a^2 + y^2/b^2 = 1$, and to find the condition that the line shall be a tangent to the ellipse.

$$y = mx + c. \qquad \qquad (3)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \qquad (4)$$

By substituting for y from (3) in (4), the abscissæ of the points of intersection are given by

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1,$$

i.e. $$x^2(b^2 + a^2m^2) + 2a^2cmx + (a^2c^2 - a^2b^2) = 0. \qquad (5)$$

The equation (5) is a quadratic in x, and hence the line (3) cuts the ellipse (4) in two points whose abscissæ are given by the equation (5). The corresponding ordinates will then be obtained from equation (3).

If the line (3) is to be a tangent to the ellipse (4), the roots of equation (5) must be coincident.

The condition for this is

$$a^4c^2m^2 = a^2(c^2 - b^2)(b^2 + a^2m^2),$$

i.e. $$a^2c^2m^2 = b^2c^2 - b^4 + a^2c^2m^2 - a^2b^2m^2,$$

i.e. $$b^2c^2 = b^2(a^2m^2 + b^2),$$

i.e. $$c^2 = a^2m^2 + b^2 \qquad \text{(since } b^2 \neq 0 \text{ and } a^2 \neq 0\text{)}.$$

Hence $$c = \pm\sqrt{(a^2m^2 + b^2)}.$$

Thus, for all values of m, the straight lines $\boldsymbol{y = mx \pm \sqrt{(a^2m^2 + b^2)}}$ are tangents to the given ellipse. This equation is known as the *slope equation* of the tangent, and shows that for any given slope m there are two tangents to an ellipse which are equidistant from the centre C.

9. Theorem.—To find the equation of the line joining the points (x_1, y_1), (x_2, y_2) on the ellipse $x^2/a^2 + y^2/b^2 = 1$, and to deduce the equation of the tangent at the point (x_1, y_1).

Since (x_1, y_1), (x_2, y_2) lie on the ellipse,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1, \qquad \qquad (6)$$

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1. \qquad \qquad (7)$$

(6) − (7) gives $$\frac{x_1^2 - x_2^2}{a^2} + \frac{y_1^2 - y_2^2}{b^2} = 0,$$

i.e. $$\frac{(x_1 - x_2)(x_1 + x_2)}{a^2} = -\frac{(y_1 - y_2)(y_1 + y_2)}{b^2}. \quad . \quad . \quad (8)$$

The equation of the line joining the points (x_1, y_1), (x_2, y_2) is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Combining (8) and (9) by multiplication, which ensures that the points (x_1, y_1), (x_2, y_2) shall lie on the ellipse, the equation of the line is

$$\frac{(x - x_1)(x_1 + x_2)}{a^2} = -\frac{(y - y_1)(y_1 + y_2)}{b^2},$$

i.e. $$\frac{x(x_1 + x_2)}{a^2} + \frac{y(y_1 + y_2)}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2}.$$

Using the result (6), the required equation is therefore

$$\frac{x(x_1 + x_2)}{a^2} + \frac{y(y_1 + y_2)}{b^2} = 1 + \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2}.$$

Replacing x_2 by x_1 and y_2 by y_1 in this equation, the equation of the tangent at (x_1, y_1) will be

$$\frac{2xx_1}{a^2} + \frac{2yy_1}{b^2} = 1 + \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}.$$

Using (6) and dividing through by 2, this becomes

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

Note.—If $x_1 = a$, and $y_1 = 0$, in this last result, the equation of the tangent at A is $x = a$, i.e. it is parallel to the directrix. Similarly, the equations of the tangents at A′, B, B′ are $x = -a$, $y = b$ and $y = -b$ respectively.

10. Theorem.—To prove that the tangents at the extremities of a chord through the centre C of an ellipse are parallel and equidistant from C.

Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If (x_1, y_1) be one extremity of a chord through C, then $(-x_1, -y_1)$ must be the other extremity.

The equations of the tangents at (x_1, y_1), $(-x_1, -y_1)$ to the ellipse (10) are respectively

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1. \qquad (11)$$

and

$$-\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1. \qquad (12)$$

From equations (11) and (12) it can be seen that the tangents are parallel. From equation (11), the length of the perpendicular from C on the tangent, neglecting sign, is $1/\sqrt{(x_1{}^2/a^4+y_1{}^2/b^4)}$; and from equation (12), the length of the perpendicular from C on the tangent (12) is $1/\sqrt{(x_1{}^2/a^4+y_1{}^2/b^4)}$, neglecting the sign. *Hence the tangents are parallel and equidistant from* C.

11. Theorem.—To find the condition that the line $lx+my+n=0$ shall touch the ellipse $x^2/a^2+y^2/b^2=1$.

$$lx+my+n=0. \qquad (13)$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad (14)$$

Let (x_1, y_1) be a point on the ellipse (14),

$$\therefore \frac{x_1{}^2}{a^2}+\frac{y_1{}^2}{b^2}=1, \qquad (15)$$

and the equation of the tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1. \qquad (16)$$

Comparing (13) and (16), which are to represent the same line,

$$\frac{x_1/a^2}{l}=\frac{y_1/b^2}{m}=-\frac{1}{n},$$

i.e.

$$x_1=-a^2l/n, \qquad (17)$$

and

$$y_1=-b^2m/n. \qquad (18)$$

Using (17) and (18) in (15), the required condition is

$$\frac{a^4l^2}{n^2a^2}+\frac{b^4m^2}{b^2n^2}=1,$$

i.e.

$$a^2l^2+b^2m^2=n^2.$$

12. Theorem.—To find the equation of the normal at the point (x_1, y_1) of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

The equation of the tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1,$$

and its slope is $(-x_1/a^2) \div (y_1/b^2)$.

Therefore the slope of the normal is $(y_1/b^2) \div (x_1/a^2)$. Hence the equation of the normal at (x_1, y_1) is

$$y - y_1 = \frac{y_1/b^2}{x_1/a^2}(x - x_1),$$

i.e.
$$\frac{y - y_1}{y_1/b^2} = \frac{x - x_1}{x_1/a^2}.$$

13. Theorem.—To prove that there are two tangents from any point (h, k) to the ellipse $x^2/a^2 + y^2/b^2 = 1$, and that these tangents are real, coincident, or imaginary, according as (h, k) lies outside, on, or inside the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The tangents of slope m to the ellipse (19) have as their equations

$$y = mx \pm \sqrt{(a^2m^2 + b^2)}. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

If the tangent passes through (h, k),

$$k = mh \pm \sqrt{(a^2m^2 + b^2)}.$$

$$\therefore\ (k - mh)^2 = a^2m^2 + b^2,$$

i.e.
$$m^2(a^2 - h^2) + 2khm + (b^2 - k^2) = 0. \quad . \quad . \quad . \quad (21)$$

The equation (21) is a quadratic in m and gives two values for m, from which it can be seen that there are two tangents from the point (h, k) to the ellipse.

Now the roots of (21) are real, coincident, or imaginary, according as

$$k^2h^2 > \text{or} = \text{or} < (a^2 - h^2)(b^2 - k^2),$$

i.e. according as

$$k^2h^2 > \text{or} = \text{or} < a^2b^2 - a^2k^2 - b^2h^2 + h^2k^2,$$

i.e. according as

$$a^2k^2 + b^2h^2 > \text{or} = \text{or} < a^2b^2,$$

i.e. according as

$$h^2/a^2 + k^2/b^2 > \text{or} = \text{or} < 1.$$

But these are the conditions that the point (h, k) shall lie outside, on, or inside the ellipse.

Hence the tangents from (h, k) to the ellipse are *real, coincident, or imaginary, according as the point* (h, k) *lies outside, on, or within the ellipse.*

14. Theorem.—To find the polar of the point (x_1, y_1) with respect to the ellipse $x^2/a^2 + y^2/b^2 = 1$.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

Let the points of contact of the tangents (real, coincident, or imaginary) from (x_1, y_1) to the ellipse (22) be (h_1, k_1), (h_2, k_2).

The equations of the tangents at these points are respectively

$$xh_1/a^2 + yk_1/b^2 = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

$$xh_2/a^2 + yk_2/b^2 = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

But the lines (23) and (24) pass through (x_1, y_1),

$$\therefore \ h_1x_1/a^2 + k_1y_1/b^2 = 1, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

$$h_2x_1/a^2 + k_2y_2/b^2 = 1. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

The equations (25) and (26) are the conditions that the points (h_1, k_1), (h_2, k_2) shall lie on the line

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1,$$

which is therefore the required equation of the polar of (x_1, y_1).

15. Theorems.—(i) If the polar of P passes through Q, then the polar of Q must pass through P. (ii) If the polars of P and Q each pass through R, then the polar of R is the line PQ.

The proofs for these two theorems are the same as in the case of the circle and parabola.

16. Theorem.—To find the locus of the points of intersection of perpendicular tangents to the ellipse $x^2/a^2 + y^2/b^2 = 1$.

The equations of the tangents of slope m to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

are given by $$y = mx \pm \sqrt{(a^2m^2 + b^2)},$$

i.e. $$a^2m^2 + b^2 = (y - mx)^2,$$

i.e. $$m^2(a^2 - x^2) + 2mxy + (b^2 - y^2) = 0. \quad . \quad . \quad . \quad (28)$$

This shows that two tangents can be drawn from any point (x, y) to the ellipse (27), since it is a quadratic in m, i.e. if the point of intersection of two tangents to the ellipse (27) be (x, y), the tangents will have slopes given by the roots of equation (28) in m.

If these two tangents be perpendicular,

$$m_1m_2 = -1.$$

From (28), $$m_1m_2 = (b^2 - y^2)/(a^2 - x^2).$$

Therefore, for perpendicular tangents,

$$(b^2 - y^2)/(a^2 - x^2) = -1,$$

i.e. the required locus is given by

$$b^2 - y^2 = -(a^2 - x^2),$$

i.e. $$x^2 + y^2 = a^2 + b^2.$$

Thus the required locus is a circle, centre C, radius $\sqrt{(a^2 + b^2)}$, and is known as the *director circle* of the ellipse.

17. Definition.—The circle described on the major axis of an ellipse as diameter is known as the auxiliary circle of the ellipse.

18. Theorem.—To prove that the corresponding ordinates of the ellipse $x^2/a^2 + y^2/b^2 = 1$ and its auxiliary circle are in the ratio $b : a$.

If P be any point (x, y) on the auxiliary circle

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and PN be the ordinate of P cutting the ellipse at P_1 in the same quadrant, then P_1 is the *corresponding point* on the ellipse to the point P on the auxiliary circle, and PN, P_1N are corresponding ordinates of the auxiliary circle and ellipse.

Since P lies on the auxiliary circle (29),

$$\frac{CN^2}{a^2} + \frac{PN^2}{a^2} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

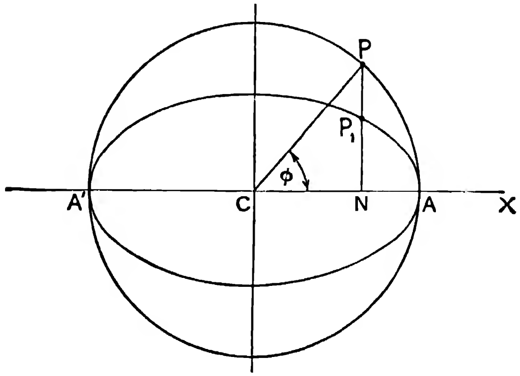
Since P_1 lies on the ellipse

$$\frac{CN^2}{a^2} + \frac{P_1N^2}{b^2} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

(30) — (31) gives

$$\frac{PN^2}{a^2} - \frac{P_1N^2}{b^2} = 0, \text{ i.e. } \frac{P_1N^2}{PN^2} = \frac{b^2}{a^2}.$$

$$\therefore \frac{P_1N}{PN} = \frac{b}{a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$



The angle PCN is known as the *eccentric angle* of the point P_1 on the ellipse.

If $\angle PCN$ be denoted by ϕ, then $CN = a \cos \phi$ and $PN = a \sin \phi$.

Therefore from the result (32) $P_1N = b \sin \phi$.

Thus, any point P_1 on the ellipse will have co-ordinates $(a \cos \phi, b \sin \phi)$, where ϕ is the eccentric angle of P_1, and the equations $x = a \cos \phi$, $y = b \sin \phi$ will be a parametric form of the equation to the ellipse $x^2/a^2 + y^2/b^2 = 1$.

The point P_1 is sometimes denoted by $[\phi]$, which means that the eccentric angle of P_1 is ϕ, i.e. $P_1 \equiv (a \cos \phi, b \sin \phi)$.

19. Theorem.—To find the equation of the chord joining the points on the ellipse $x^2/a^2 + y^2/b^2 = 1$, whose eccentric angles are ϕ_1 and ϕ_2.

The points whose eccentric angles are ϕ_1 and ϕ_2 are given by $(a \cos \phi_1, b \sin \phi_1)$, $(a \cos \phi_2, b \sin \phi_2)$ respectively. Hence the chord joining the points is represented by the equation

$$\frac{x - a \cos \phi_1}{a \cos \phi_1 - a \cos \phi_2} = \frac{y - b \sin \phi_1}{b \sin \phi_1 - b \sin \phi_2},$$

i.e. $(x - a\cos\phi_1)\,2b\cos\tfrac{1}{2}(\phi_1 + \phi_2)\sin\tfrac{1}{2}(\phi_1 - \phi_2)$

$$= (y - b\sin\phi_1)\,2a\sin\tfrac{1}{2}(\phi_1 + \phi_2)\sin\tfrac{1}{2}(\phi_2 - \phi_1),$$

i.e. $b\cos\tfrac{1}{2}(\phi_1 + \phi_2)(x - a\cos\phi_1) = -a\sin\tfrac{1}{2}(\phi_1 + \phi_2)(y - b\sin\phi_1)$,

i.e. $bx\cos\tfrac{1}{2}(\phi_1 + \phi_2) + ay\sin\tfrac{1}{2}(\phi_1 + \phi_2)$

$$= ab[\cos\phi_1\cos\tfrac{1}{2}(\phi_1 + \phi_2) + \sin\phi_1\sin\tfrac{1}{2}(\phi_1 + \phi_2)],$$

i.e. $(x/a)\cos\tfrac{1}{2}(\phi_1 + \phi_2) + (y/b)\sin\tfrac{1}{2}(\phi_1 + \phi_2) = \cos\tfrac{1}{2}(\phi_1 - \phi_2)$.

Using $\phi_1 = \phi_2 = \phi$ in this result, it follows that the equation of the tangent at the point $[\phi]$ is

$$(x/a)\cos\phi + (y/b)\sin\phi = 1.$$

This result could also be obtained from the equation to the tangent at the point (x_1, y_1), which is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1,$$

by replacing x_1 and y_1 by $a\cos\phi$ and $b\sin\phi$ respectively.

From the equation to the chord, namely,

$$(x/a)\cos\tfrac{1}{2}(\phi_1 + \phi_2) + (y/b)\sin\tfrac{1}{2}(\phi_1 + \phi_2) = \cos\tfrac{1}{2}(\phi_1 - \phi_2),$$

it can be seen that, for chords having the same slope (i.e. a system of parallel chords), the value of $(-b/a)\cot\tfrac{1}{2}(\phi_1 + \phi_2)$ is constant, and therefore $\phi_1 + \phi_2 =$ constant.

Conversely, if $(\phi_1 + \phi_2)$ be constant, then the chords are parallel.

20. Theorem.—To find the equation of the normal at the point whose eccentric angle is ϕ.

The equation of the tangent at the point $(a\cos\phi, b\sin\phi)$ is

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1.$$

Its slope is $-(b/a)\cot\phi$ and therefore the slope of the normal at the point is $(a/b)\tan\phi$.

Hence the equation of the normal at the point $[\phi]$ is

$$y - b\sin\phi = (a/b)\tan\phi(x - a\cos\phi),$$

i.e. $$\frac{b(y - b\sin\phi)}{\sin\phi} = \frac{a(x - a\cos\phi)}{\cos\phi},$$

i.e. $$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2.$$

21. Theorem.—To find the locus of the middle points of a system of parallel chords of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Let each of the system of parallel chords have slope m, and let $P \equiv [\phi_1]$, $Q \equiv [\phi_2]$ be the extremities of one of the chords of the system.

Let $(\bar{x}, \bar{y})$ be the mid-point of PQ.

$$\text{Then } \bar{x} = \tfrac{1}{2}(a\cos\phi_1 + a\cos\phi_2) = a\cos\tfrac{1}{2}(\phi_1+\phi_2)\cos\tfrac{1}{2}(\phi_1-\phi_2),$$

$$\bar{y} = \tfrac{1}{2}(b\sin\phi_1 + b\sin\phi_2) = b\sin\tfrac{1}{2}(\phi_1+\phi_2)\cos\tfrac{1}{2}(\phi_1-\phi_2).$$

$$\therefore \frac{\bar{y}}{\bar{x}} = \frac{b}{a}\tan\tfrac{1}{2}(\phi_1+\phi_2). \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The equation of PQ has been shown to be

$$\frac{x}{a}\cos\tfrac{1}{2}(\phi_1+\phi_2) + \frac{y}{b}\sin\tfrac{1}{2}(\phi_1+\phi_2) = \cos\tfrac{1}{2}(\phi_1-\phi_2),$$

and $$\therefore m = -\frac{b}{a}\cot\tfrac{1}{2}(\phi_1+\phi_2). \quad . \quad . \quad . \quad . \quad (34)$$

From (33) × (34), $$\frac{m\bar{y}}{\bar{x}} = -\frac{b^2}{a^2},$$

i.e. $$\bar{y} = -\frac{b^2}{a^2m}\bar{x}.$$

Hence the locus of the mid-points of parallel chords of the system is

$$y = -\frac{b^2}{a^2m}x.$$

This is a diameter of the ellipse, since it passes through C.

If m_1 be the slope of the diameter bisecting chords of a system whose slopes are m, it follows that

$$mm_1 = -b^2/a^2.$$

From the symmetry of this result it follows that, if the line $y = mx$ bisect all chords parallel to the diameter $y = m_1x$, then the diameter $y = m_1x$ will bisect all chords parallel to the diameter $y = mx$.

22. Definition.—Two diameters are said to be conjugate when each bisects all chords parallel to the other diameter.

Hence the two diameters $y = m_1x$, $y = m_2x$ are conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$, if $m_1m_2 = -b^2/a^2$.

Note.—Since the tangent is the limiting position of a chord, it follows that the tangent at an extremity of a diameter is parallel to the system of chords bisected by that diameter.

23. Theorem.—To prove that the eccentric angles of the extremities of two conjugate diameters differ by $\frac{1}{2}\pi$.

Let $P \equiv [\phi_1]$ be an extremity of one diameter, and $V \equiv [\phi_2]$ be an extremity of the conjugate diameter.

The slope of CP will be $(b \tan \phi_1)/a$, and the slope of CV will be $(b \tan \phi_2)/a$.

Since the diameters are conjugate the product of the slopes is $-b^2/a^2$.

$$\therefore \quad \frac{b^2}{a^2} \tan \phi_1 \tan \phi_2 = -\frac{b^2}{a^2},$$

i.e. $$\tan \phi_1 \tan \phi_2 = -1, \quad \therefore \ \varphi_1 \sim \varphi_2 = \tfrac{1}{2}\pi.$$

24. Theorem.—To prove that, if P and V be the extremities of conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$, then $CP^2 + CV^2 = a^2 + b^2$.

Let $P \equiv [\phi_1]$ and $V \equiv [\phi_2]$, then $\phi_2 = \phi_1 \pm \frac{1}{2}\pi$.

Now $\quad P \equiv [a \cos \phi_1, b \sin \phi_1]$, $V \equiv [a \cos (\phi_1 \pm \frac{1}{2}\pi), b \sin (\phi_1 \pm \frac{1}{2}\pi)]$.

$$CP^2 = a^2 \cos^2 \phi_1 + b^2 \sin^2 \phi_1.$$

$$\begin{aligned} CV^2 &= a^2 \cos^2 (\phi_1 \pm \tfrac{1}{2}\pi) + b^2 \sin^2 (\phi_1 \pm \tfrac{1}{2}\pi) \\ &= a^2 \sin^2 \phi_1 + b^2 \cos^2 \phi_1. \end{aligned}$$

$$\begin{aligned} \therefore CP^2 + CV^2 &= (a^2 \cos^2 \phi_1 + b^2 \sin^2 \phi_1) + (a^2 \sin^2 \phi_1 + b^2 \cos^2 \phi_1) \\ &= (a^2 + b^2)(\cos^2 \phi_1 + \sin^2 \phi_1) \\ &= a^2 + b^2. \end{aligned}$$

25. Theorem.—To prove that the area of the parallelogram touching the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the ends of two conjugate diameters is constant and equal to $4ab$.

Since the tangents at the extremities of a diameter are parallel to the conjugate diameter, it follows that the parallelogram formed by the four tangents is divided into four equal parallelograms by the conjugate diameters.

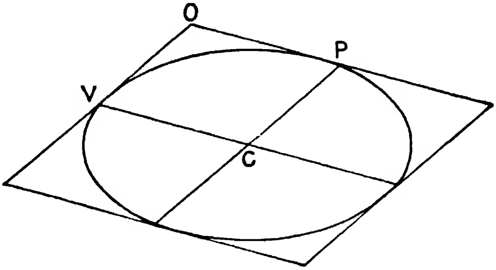
Hence the area of the required parallelogram is four times the area of one of the small parallelograms shown in the figure.

Let P and V be two extremities of conjugate diameters as shown in the diagram, and the completed small parallelogram be CPOV, where C is the centre of the ellipse.

Let $$\mathrm{P} \equiv [\phi_1], \quad \therefore \mathrm{V} \equiv [\phi_1 \pm \tfrac{1}{2}\pi].$$

As proved in the previous theorem,

$$\mathrm{CV}^2 = a^2 \sin^2 \phi_1 + b^2 \cos^2 \phi_1.$$



The equation of the tangent at P is

$$\frac{x}{a} \cos \phi_1 + \frac{y}{b} \sin \phi_1 = 1.$$

The length (p) of the perpendicular from C on OP is given by

$$p = 1 \Big/ \sqrt{\left(\frac{\cos^2 \phi_1}{a^2} + \frac{\sin^2 \phi_1}{b^2}\right)}$$
$$\frac{ab}{\sqrt{(a^2 \sin^2 \phi_1 + b^2 \cos^2 \phi_1)}}$$

The area of the parallelogram CPOV $= p \,.\, \mathrm{CV}$

$$= \frac{ab}{\sqrt{(a^2 \sin^2 \phi_1 + b^2 \cos^2 \phi_1)}} \times \sqrt{(a^2 \sin^2 \phi_1 + b^2 \cos^2 \phi_1)}$$
$$= ab.$$

Hence *the area of the parallelogram formed by the four tangents is* $4ab$.

Corollary.—If CP $= r_1$, CV $= r_2$, and θ be the acute angle between CP and CV, then $r_1 r_2 \sin \theta = ab$. Now θ is least when $\sin \theta$ is least. Hence from this result it can be seen that θ is least when $r_1 r_2$ is greatest.

But $r_1{}^2 + r_2{}^2 = a^2 + b^2$, i.e. $(r_1 - r_2)^2 + 2r_1 r_2 = a^2 + b^2$,

i.e. $$2r_1 r_2 = a^2 + b^2 - (r_1 - r_2)^2.$$

From this it can be seen that $r_1 r_2$ is greatest when $r_1 = r_2$. *Hence* θ

is least when the conjugate diameters are of equal length, i.e. for equi-conjugate diameters.

26. Theorem.—To find the equations of the equi-conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Using the notation of the previous theorem, with $P \equiv [\phi_1]$ and therefore $V \equiv [\phi_1 \pm \frac{1}{2}\pi]$,

$$CP^2 = a^2 \cos^2 \phi_1 + b^2 \sin^2 \phi_1.$$

$$CV^2 = a^2 \sin^2 \phi_1 + b^2 \cos^2 \phi_1.$$

$$\therefore\ CP^2 - CV^2 = (a^2 - b^2)(\cos^2 \phi_1 - \sin^2 \phi_1) = (a^2 - b^2) \cos 2\phi_1.$$

$$\therefore\ CP = CV, \text{ when } \cos 2\phi_1 = 0,$$

i.e. when $2\phi_1 = \frac{1}{2}\pi, \frac{3}{2}\pi$, etc., $\therefore\ \phi_1 = \frac{1}{4}\pi$ or $\frac{3}{4}\pi$. $(\phi_1 \not> \pi)$.

In this case

$$P \equiv [a \cos \tfrac{1}{4}\pi, b \sin \tfrac{1}{4}\pi] \text{ or } [a \cos \tfrac{3}{4}\pi, b \sin \tfrac{3}{4}\pi].$$

Thus the equation of CP is

$$\frac{y}{x} = \frac{b}{a} \tan \tfrac{1}{4}\pi, \text{ or } \frac{y}{x} = \frac{b}{a} \tan \tfrac{3}{4}\pi,$$

i.e.

$$\frac{y}{x} = \frac{b}{a}, \text{ or } \frac{y}{x} = -\frac{b}{a}.$$

Hence the equations of the equi-conjugate diameters are given by

$$y = \pm \frac{b}{a} x.$$

Note.—These lines are the diagonals of the rectangle formed by the tangents at the extremities of the two axes.

27. Definition.—The two lines formed by joining any point on an ellipse to the extremities of a diameter are known as *supplemental chords*.

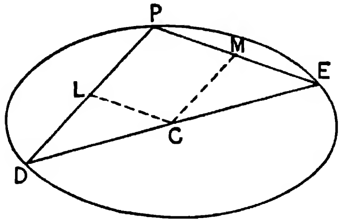
28. Theorem.—To prove that two supplemental chords are parallel to a pair of conjugate diameters.

Let P be any point on the ellipse centre C, and DE any diameter of the ellipse. L and M are the mid-points of PD and PE respectively.

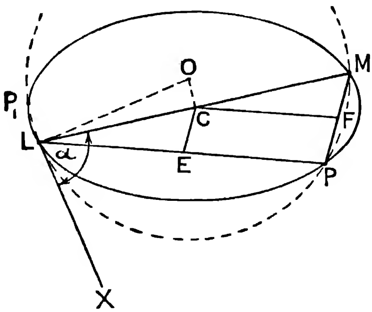
By geometry, CM is parallel to PD, and CL is parallel to PE, thus CM is a portion of a diameter bisecting a chord parallel to the diameter of which CL is a part, and vice versa.

Hence CL and CM are conjugate diameters.

Therefore PD and PE are parallel to a pair of conjugate diameters.



This result gives a method of constructing two conjugate diameters at a given angle.



Let LM be any diameter of the ellipse whose centre is C and let the angle between the required conjugate diameters be α. With LM as chord, construct a segment of a circle to contain the given angle α intersecting the ellipse at P and P_1. Join PL and PM, and take E and F the mid-points of PL and PM respectively. Then the required conjugate diameters will be given by the lines CE and CF produced.

29. Theorem.—To prove that the product of the focal radii at any point of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is equal to the square on the semi-diameter parallel to the tangent at the point.

Let $P \equiv (x_1, y_1)$ be the point, C the centre of the ellipse, S and S′ its foci, etc.

It has been shown that

$$SP = a - ex_1, \quad S'P = a + ex_1.$$

$$\therefore\ SP \,.\, S'P = a^2 - e^2x_1^2.$$

The tangent at P is parallel to the conjugate diameter of the diameter PC produced. Hence if CV be the semi-diameter parallel to the tangent at P, then CP and CV are conjugate semi-diameters.

Thus, if $P \equiv [\phi_1]$, then $V \equiv [\phi_1 \pm \frac{1}{2}\pi]$,

and
$$\begin{aligned} CV^2 &= a^2 \sin^2 \phi_1 + b^2 \cos^2 \phi_1 \\ &= a^2 y_1^2/b^2 + b^2 x_1^2/a^2 \\ &= a^2(1 - x_1^2/a^2) + a^2(1 - e^2)x_1^2/a^2 \\ &\qquad \{\text{using } x_1^2/a^2 + y_1^2/b^2 = 1, \text{ and } b^2 = a^2(1 - e^2)\} \\ &= a^2 - x_1^2 + x_1^2 - e^2 x_1^2 \\ &= a^2 - e^2 x_1^2 = SP \,.\, S'P. \end{aligned}$$

30. Theorem.—To find the polar equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$, referred to its centre as pole and OX as its initial line.

If (r, θ) be the polar co-ordinates of any point $P \equiv (x, y)$ on the ellipse, then

$$x = r\cos\theta,\ y = r\sin\theta.$$

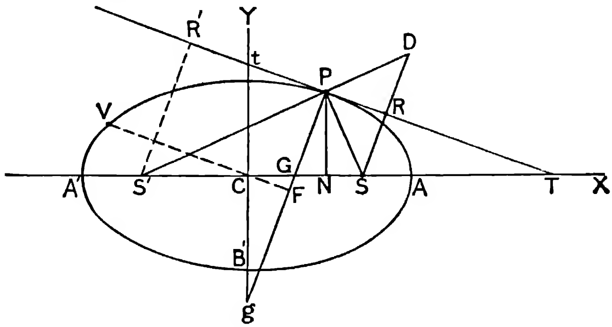
Using these in the equation of the ellipse, the polar form of the equation is

$$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}.$$

Geometrical Properties of an Ellipse

31. The points not already stated in the diagram shown are as follows:

PT is the tangent at P meeting CX in T and CY in t. SR and S′R′ are the perpendiculars from S and S′ on the tangent at P. SR



is produced to meet S′P produced at D. PG is the normal at P meeting CX in G and CY in g. CF is a line parallel to PT meeting Pg at F, and the ellipse $x^2/a^2 + y^2/b^2 = 1$ at V.

32. Theorem.—To prove that CN . CT $= a^2$.

Let P be the point (x_1, y_1) on the ellipse $x^2/a^2 + y^2/b^2 = 1$. The equation of the tangent at P is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

At T, where $y = 0$, $\quad xx_1/a^2 = 1$, i.e. $xx_1 = a^2$.

$$\therefore \ \mathbf{CN} \,.\, \mathbf{CT} = a^2.$$

33. Theorem.—To prove that PN . Ct $= b^2$.

With P $\equiv (x_1, y_1)$, the equation of the tangent at P is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

At t, where this line cuts CY, $x = 0$;

$$\therefore \ yy_1/b^2 = 1, \text{ i.e. } yy_1 = b^2.$$
$$\therefore \ \mathbf{PN} \,.\, \mathbf{Ct} = b^2.$$

34. Theorem.—To prove that, if P $\equiv (x_1, y_1)$, then CG $= e^2x_1$, and PG bisects the angle SPS′.

The equation of the normal at P is

$$\frac{x - x_1}{x_1/a^2} = \frac{y - y_1}{y_1/b^2}.$$

Where this normal cuts CX, i.e. at G, $y = 0$, and hence

$$\frac{x - x_1}{x_1/a^2} = -y_1\Big/\frac{y_1}{b^2} = -b^2.$$

$$\therefore \ x = x_1 - \frac{b^2}{a^2}x_1 = \frac{x_1}{a^2}\{a^2 - a^2(1 - e^2)\}$$
$$= e^2x_1.$$

Now $\qquad \text{GS} = \text{CS} - \text{CG} = ae - e^2x_1 = e(a - ex_1),$

and $\qquad \text{GS}' = \text{CS}' + \text{CG} = ae + e^2x_1 = e(a + ex_1).$

$$\therefore \ \frac{\text{S}'\text{G}}{\text{GS}} = \frac{a + ex_1}{a - ex_1}.$$

But it has been proved that

$$\text{SP} = a - ex_1, \text{ and } \text{S'P} = a + ex_1,$$

$$\therefore \frac{\text{S'P}}{\text{SP}} = \frac{a + ex_1}{a - ex_1} = \frac{\text{S'G}}{\text{GS}},$$

$$\therefore \frac{\text{S'G}}{\text{GS}} = \frac{\text{S'P}}{\text{SP}}, \quad \therefore \textbf{GP bisects } \angle\textbf{SPS}'.$$

Since PT is perpendicular to PG, it follows that PT *bisects* $\angle$SPS' *externally.*

35. Theorem.—To prove that PF . PG = b^2 for the ellipse

$$x^2/a^2 + y^2/b^2 = 1.$$

Now G $\equiv$ $(e^2x_1, 0)$, using P $\equiv$ (x_1, y_1),

$$\therefore \text{PG} = \sqrt{\{(x_1 - e^2x_1)^2 + y_1^2\}}$$

$$= \sqrt{\{x_1^2(1 - e^2)^2 + y_1^2\}} = \sqrt{\left\{x_1^2 \frac{b^4}{a^4} + y_1^2\right.}$$

$$= b^2 \sqrt{\left\{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}\right\}}.$$

Now PF is equal in length to the perpendicular from C on the tangent PT, and the equation of PT is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1,$$

therefore length of the perpendicular from C on PT is

$$1\Big/\sqrt{\left\{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}\right\}} = \text{PF}.$$

$$\therefore \textbf{PG . PF} = b^2.$$

36. Theorem.—To prove that SR . S'R' = b^2.

The equation of the tangent PT of slope m is

$$y = mx + \sqrt{(a^2m^2 + b^2)} \qquad \text{(using +ve sign).}$$

Therefore the perpendicular from S on PT is SR given by

$$\text{SR} = \frac{aem + \sqrt{(a^2m^2 + b^2)}}{\sqrt{(1 + m^2)}} \quad [\text{since S} \equiv (ae, 0)].$$

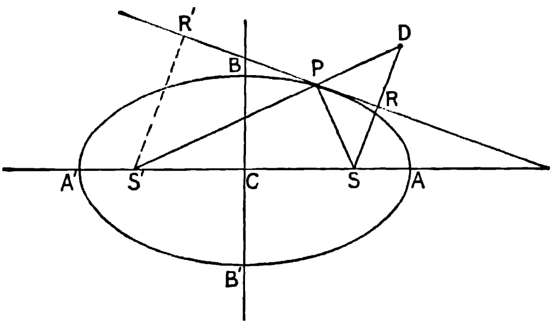
Since $S' \equiv (-ae, 0)$, it follows similarly that

$$S'R' = \frac{-aem + \sqrt{(a^2m^2 + b^2)}}{\sqrt{(1 + m^2)}}.$$

$$\begin{aligned}\therefore\ SR \,.\, S'R' &= \{\sqrt{(a^2m^2 + b^2)} + aem\}\{\sqrt{(a^2m^2 + b^2)} - aem\}/(1 + m^2) \\ &= (a^2m^2 + b^2 - a^2e^2m^2)/(1 + m^2) \\ &= \{m^2a^2(1 - e^2) + b^2\}/(1 + m^2) \\ &= \frac{b^2m^2 + b^2}{1 + m^2} = b^2.\end{aligned}$$

37. Theorem.—To prove that the locus of R or R′ is the auxiliary circle of the ellipse.

Proof (i).—Since PR bisects $\angle SPD$ and also is perpendicular to the line SD, it must bisect SD, and also $PS = PD$; thus R is the mid-point of SD, and $S'D = S'P + PD = S'P + PS = 2a$. Now C is



the mid-point of SS′, and R is the mid-point of SD, therefore $CR = \frac{1}{2}S'D = a$. But C is a fixed point, therefore the locus of R is the circle, centre C, radius a, i.e. the auxiliary circle of the ellipse.

Proof (ii) (analytical method).—Let the tangent at P be of slope m, and its equation will therefore be (taking +ve sign)

$$y = mx + \sqrt{(a^2m^2 + b^2)},$$

i.e. $$y - mx = \sqrt{(a^2m^2 + b^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (35)$$

The slope of SR will be $-1/m$, since SR is perpendicular to PT. Thus the equation of SR is

$$y = -\frac{1}{m}(x - ae),$$

i.e. $$my + x = ae. \quad . \qquad (36)$$

$(35)^2 + (36)^2$ gives, for the point of intersection of lines (35) and (36), i.e. R,

$$y^2(1 + m^2) + x^2(1 + m^2) = a^2m^2 + b^2 + a^2e^2$$
$$= a^2m^2 + a^2(1 - e^2) + a^2e^2$$
$$= a^2(1 + m^2),$$

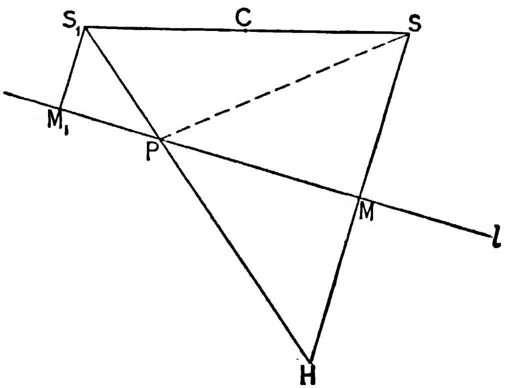
i.e.
$$x^2 + y^2 = a^2 \qquad \text{(since } 1 + m^2 \neq 0\text{).}$$

This is the equation of the locus of R and is the auxiliary circle.

Example 1 (L.U.).—C is the mid-point of SS_1 and M is the foot of the perpendicular from S to any straight line l coplanar with SS_1. SM is produced to H so that SM = MH, and the join of S_1 to H intersects the line l at P. If M describes a circle whose centre is C, show that

(i) the locus of P is an ellipse;

(ii) if M_1 be the foot of the perpendicular from S_1 to the line l, then $SM . S_1M_1$ is constant.



(i) Join PS.

By geometry,
$$PS = PH.$$
$$\therefore\ PS + PS_1 = PH + PS_1$$
$$= HS_1.$$

Since C and M are the mid-points of SS_1 and SH,
$$HS_1 = 2CM.$$

Therefore HS_1 is constant (M describes circle centre C and therefore CM is constant),

i.e.
$$SP + S_1P = \text{constant}.$$

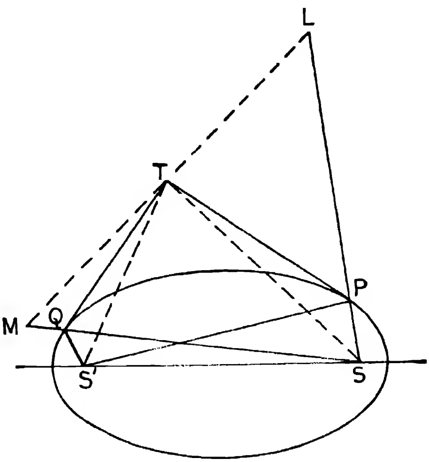
Thus P describes an ellipse with S and S_1 as foci and centre C.

(ii) By geometry, PM bisects $\angle$SPH, therefore PM must be a tangent to the ellipse traced out by P.

$$\therefore \ SM \, . \, S_1M_1 = \text{square on minor axis of ellipse} = \text{constant.}$$

Example 2 (L.U.).—If a pair of tangents TP, TQ be drawn to an ellipse from a point T, and S is a focus, prove that SP, SQ make equal angles with ST.

PSP′ is a variable focal chord of an ellipse, S being a focus. Tangents to the ellipse at P and P′ meet a fixed tangent l at Q and Q′ respectively. Show that QQ′ subtends a right angle at S, and deduce that the circles on QQ′ as diameter belong to an intersecting system of coaxal circles.



S′ is the other focus and SP is produced to L so that PL = PS′. SQ is produced to M so that QM = S′Q.

Now PT bisects $\angle$LPS′ (theorem), therefore in triangles TLP, TPS′,

$$\angle LPT = \angle TPS' \qquad \text{(proved)},$$

$$PT \text{ is common},$$

$$PL = S'P \qquad \text{(constr.)}.$$

$$\therefore \ \triangle TLP \equiv \triangle TPS',$$

$$\therefore \ TL = TS'.$$

Similarly, $\triangle TQM \equiv \triangle TQS'$ (2 sides and included $\angle$ equal),

$$\therefore \ TM = TS';$$

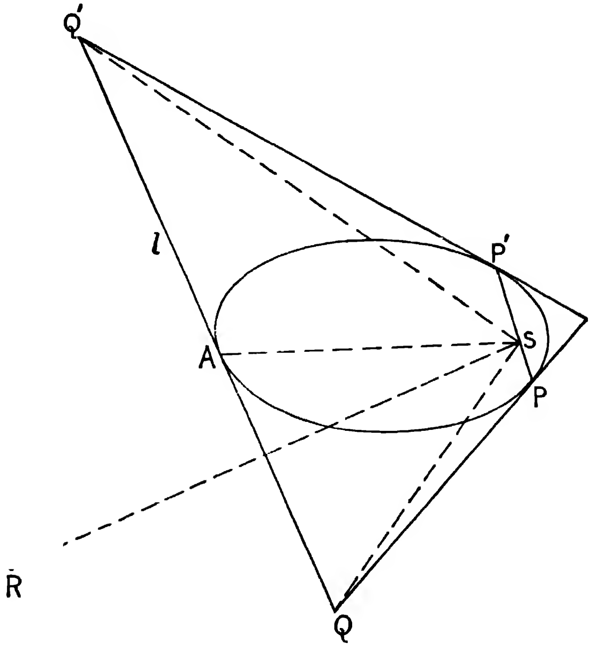
$$\therefore \ TL = TM.$$

Now $\quad SP + PL = SP + S'P = SQ + S'Q = SQ + QM = SM$

$$(SP + S'P = SQ + S'Q = 2a).$$

Therefore in triangles TSL and TSM,

$$\begin{aligned} SL &= SM && \text{(proved),} \\ TS &\text{ is common,} \\ TL &= TM && \text{(proved).} \\ \therefore\ \triangle TSL &\equiv \triangle TSM && \text{(3 sides equal),} \\ \therefore\ \angle TSP &= \angle TSQ. \end{aligned}$$



Let A be the point of contact of the fixed tangent l. Join S to Q, A, and Q′. From the first part of the question,

$$\angle QSA = \tfrac{1}{2}\angle PSA \quad \text{and} \quad \angle ASQ' = \tfrac{1}{2}\angle ASP',$$

$$\therefore\ \angle QSQ' = \angle QSA + \angle ASQ' = \tfrac{1}{2}(\angle PSA + \angle ASP')$$

$$= \tfrac{1}{2}\text{ straight angle} = 90^\circ.$$

All circles on QQ′ as diameter will pass through the fixed point S and also through the fixed point R which is the image of S in line l. Therefore, since they all pass through two fixed points, they belong to an intersecting system of coaxal circles.

Example 3 (L.U.).—Find the co-ordinates of P, the pole of the line $lx + my = 1$, with respect to the ellipse whose equation is $b^2x^2 + a^2y^2 = a^2b^2$. Deduce, or find by other means, the condition that the line and ellipse shall touch.

O is the centre of the above ellipse, and M and N are the feet of the perpendiculars from O and P respectively on to the polar of P. If OM . PN $= \lambda$, where λ is constant, show that the polar touches the ellipse whose equation is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

Let the pole of $\qquad lx + my = 1, \quad \ldots \quad$ (i)

with respect to the ellipse $\qquad b^2x^2 + a^2y^2 = a^2b^2, \quad \ldots \quad$ (ii)

be the point $P \equiv (x_1, y_1)$.

The polar of P with respect to the ellipse (ii) is

$$b^2xx_1 + a^2yy_1 = a^2b^2. \quad \ldots \quad \text{(iii)}$$

The equations (i) and (iii) must be equivalent, therefore, comparing coefficients

$$b^2x_1/l = a^2y_1/m = a^2b^2,$$

$$\therefore\ x_1 = la^2,\ y_1 = mb^2,$$

i.e. $$P \equiv (la^2, mb^2).$$

If the line (i) touch the ellipse (ii), the point P must lie on the ellipse (ii), for the polar of a point on the curve is the tangent at the point,

i.e. $$b^2(la^2)^2 + a^2(mb^2)^2 = a^2b^2,$$

i.e. $$l^2a^2 + m^2b^2 = 1. \quad \ldots \quad \text{(iv)}$$

Using the formula for the length of the perpendicular from (x_1, y_1) on the line $ax + by + c = 0$, namely $\pm\dfrac{ax_1 + by_1 + c}{\sqrt{(a^2+b^2)}}$,

$$\text{OM} = \frac{1}{\sqrt{(l^2+m^2)}}, \quad \text{PN} = \frac{-a^2l^2 - b^2m^2 + 1}{\sqrt{(l^2+m^2)}}$$

(perpendicular from O on any line is positive).

$$\therefore\ \text{OM . PN} = (-a^2l^2 - b^2m^2 + 1)/(l^2 + m^2) = \lambda,$$

$$\therefore\ -a^2l^2 - b^2m^2 + 1 = \lambda(l^2 + m^2),$$

$$\therefore\ l^2(a^2 + \lambda) + m^2(b^2 + \lambda) = 1.$$

From the result (iv) this is the condition that the line (i) shall touch the ellipse

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

Hence the polar touches the ellipse $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1.$

Example 4 (L.U.).—Show that the equation of the chord of the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2,$$

whose mid-point P is (α, β), is

$$\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}.$$

Show that, if the pole of this chord lie on the circle $x^2 + y^2 = a^2$, then P is on the curve $a^2b^4(x^2 + y^2) = (b^2x^2 + a^2y^2)^2$, and find the point P corresponding to the point $(a/\sqrt{2}, a/\sqrt{2})$ on the circle.

Any chord of the ellipse $\quad b^2x^2 + a^2y^2 = a^2b^2, \quad \dots \dots \dots$ (i)

having $P \equiv (\alpha, \beta)$ lying on it, will have as its equation

$$y - \beta = m(x - \alpha),$$

i.e.
$$y = \beta + m(x - \alpha). \qquad \dots \dots \dots \text{(ii)}$$

Substituting from (ii) in (i) for the abscissæ of the points of intersection,

$$b^2x^2 + a^2\{\beta^2 + 2\beta m(x - \alpha) + m^2(x - \alpha)^2\} = a^2b^2,$$

i.e.
$$x^2\{b^2 + a^2m^2\} + 2x\{a^2\beta m - a^2\alpha m^2\}$$
$$+ \{a^2\beta^2 - 2a^2\alpha\beta m + a^2\alpha^2m^2 - a^2b^2\} = 0. \qquad \dots \text{(iii)}$$

If x_1, x_2 be the roots of (iii), then

$$\frac{x_1 + x_2}{2} = -\frac{(a^2\beta m - a^2\alpha m^2)}{b^2 + a^2m^2}.$$

If (α, β) be the mid-point of the chord,

$$\frac{x_1 + x_2}{2} = \alpha.$$

$$\therefore \; -(a^2\beta m - a^2\alpha m^2)/(b^2 + a^2m^2) = \alpha,$$

i.e.
$$a^2\alpha m^2 - a^2\beta m = \alpha b^2 + a^2\alpha m^2,$$

i.e.
$$-a^2\beta m = \alpha b^2, \quad \therefore \; m = -\alpha b^2/(\beta a^2).$$

Hence the equation of the chord bisected at (α, β) is

$$y - \beta = -\frac{\alpha b^2}{\beta a^2}(x - \alpha),$$

i.e.
$$\frac{\beta y - \beta^2}{b^2} = \frac{(-\alpha x + \alpha^2)}{a^2},$$

i.e.
$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}. \qquad \dots \dots \dots \text{(iv)}$$

Let (x_1, y_1) be the pole of this chord (iv) with respect to the ellipse (i), then

$$b^2xx_1 + a^2yy_1 = a^2b^2 \qquad \dots \dots \dots \text{(v)}$$

will represent the polar of (x_1, y_1), and equations (iv) and (v) will be equivalent.

Comparing coefficients in (iv) and (v),

$$\frac{b^2x_1}{\alpha/a^2} = \frac{a^2y_1}{\beta/b^2} = \frac{a^2b^2}{\alpha^2/a^2 + \beta^2/b^2},$$

i.e.
$$x_1 = \frac{\alpha}{\alpha^2/a^2 + \beta^2/b^2}, \text{ and } y_1 = \frac{\beta}{\alpha^2/a^2 + \beta^2/b^2}. \qquad \dots \text{(vi)}$$

Since (x_1, y_1) lies on the circle $x^2 + y^2 = a^2$, using equations (vi),

$$\frac{\alpha^2}{(\alpha^2/a^2 + \beta^2/b^2)^2} + \frac{\beta^2}{(\alpha^2/a^2 + \beta^2/b^2)^2} = a^2,$$

i.e.
$$\alpha^2 + \beta^2 = \frac{a^2}{a^4b^4}\{\alpha^2b^2 + \beta^2a^2\}^2.$$

Hence the point $P \equiv (\alpha, \beta)$ will lie on the curve

$$a^2b^4(x^2 + y^2) = (b^2x^2 + a^2y^2)^2.$$

From (vi), when $x_1 = a/\surd 2$ and $y_1 = a/\surd 2$,

$$\frac{a}{\surd 2} = \frac{\alpha}{\alpha^2/a^2 + \beta^2/b^2} = \frac{\beta}{\alpha^2/a^2 + \beta^2/b^2}.$$

From these equations $\alpha = \beta$ and

$$\therefore \frac{a}{\surd 2} = \frac{\alpha}{\alpha^2/a^2 + \alpha^2/b^2}, \quad \text{i.e. } a\left\{\frac{\alpha}{a^2} + \frac{\alpha}{b^2}\right\} = \surd 2.$$

$$\therefore \alpha\left\{\frac{a^2 + b^2}{ab^2}\right\} = \surd 2, \quad \therefore \alpha = \beta = \frac{\surd 2 \,.\, ab^2}{a^2 + b^2}.$$

Thus the required point is

$$\left(\frac{\surd 2 \,.\, ab^2}{a^2 + b^2}, \frac{\surd 2 \,.\, ab^2}{a^2 + b^2}\right).$$

Example 5 (L.U.).—Show that the equation of the chord joining the points on the ellipse $x^2/a^2 + y^2/b^2 = 1$, whose eccentric angles are θ and ϕ, is

$$(x/a) \cos \tfrac{1}{2}(\theta + \phi) + (y/b) \sin \tfrac{1}{2}(\theta + \phi) = \cos \tfrac{1}{2}(\theta - \phi).$$

If this chord touch a circle described on the minor axis of the ellipse as diameter, prove that its length is $a \sin (\theta - \phi)$.

The first part of the question has been covered in the bookwork.
The circle on the minor axis as diameter is

$$x^2 + y^2 = b^2, \quad \dots\dots\dots \quad \text{(i)}$$

and the perpendicular from its centre O on the given chord is

$$\frac{\cos \tfrac{1}{2}(\theta - \phi)}{\surd\{(1/a^2) \cos^2 \tfrac{1}{2}(\theta + \phi) + (1/b^2) \sin^2 \tfrac{1}{2}(\theta + \phi)\}},$$

which must equal b if the chord is to touch circle (i),

i.e.
$$a \cos \tfrac{1}{2}(\theta - \phi) = \surd\{b^2 \cos^2 \tfrac{1}{2}(\theta + \phi) + a^2 \sin^2 \tfrac{1}{2}(\theta + \phi)\}. \quad . \quad \text{(ii)}$$

The length of the chord joining the points

$$(a \cos \theta, b \sin \theta) \text{ and } (a \cos \phi, b \sin \phi),$$

whose eccentric angles are θ and ϕ, is

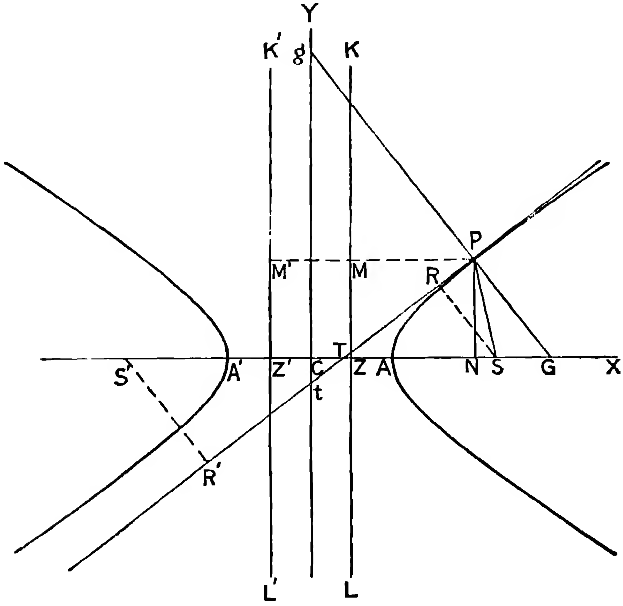
$$\sqrt{\{a^2(\cos\theta - \cos\phi)^2 + b^2(\sin\theta - \sin\phi)^2\}}$$
$$= \sqrt{\{a^2[2\sin\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\phi-\theta)]^2 + b^2[2\cos\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\theta-\phi)]^2\}}$$
$$= \sqrt{\{4\sin^2\tfrac{1}{2}(\theta-\phi)[a^2\sin^2\tfrac{1}{2}(\theta+\phi) + b^2\cos^2\tfrac{1}{2}(\theta+\phi)]\}}$$
$$= 2\sin\tfrac{1}{2}(\theta-\phi) \times a\cos\tfrac{1}{2}(\theta-\phi) \qquad \text{[using (ii)]}$$
$$= a\sin(\theta-\phi).$$

$$\therefore \text{ Length of chord} = a\sin(\theta-\phi).$$

The Hyperbola

38. Definition.—The hyperbola is a conic whose eccentricity is greater than unity, i.e. the locus of a point that moves in a plane so that its distance from a fixed point in the plane bears a constant ratio, greater than unity, to its distance from a fixed straight line in the plane.

39. Theorem.—To find, with the most convenient choice of origin and axes, the simplest form (canonical form) of the equation of a hyperbola.



In the diagram, S is the focus, KL the directrix, SZ is the perpendicular from S on KL, and SZ produced is chosen as the x-axis. A

and A′ are points dividing SZ internally and externally respectively in the ratio $e:1$, where e is the eccentricity of the hyperbola, and therefore A and A′ lie on the hyperbola. Let $AA' = 2a$ and C, the mid-point of AA′, be chosen as the origin of co-ordinates with CY perpendicular to CS as the y-axis.

$P \equiv (x, y)$ is any point on the curve, PN its ordinate and PM the perpendicular from P on KL. Z′ is the image of Z in the axis CY and S′ is the image of S in the axis CY. K′L′ is parallel to KL through Z′, and PT is the tangent at P, meeting CX in T and CY in t. SR, S′R′ are the perpendiculars on PT from S and S′ respectively. PM produced meets K′L′ in M′, and PG is the normal at P, meeting CX at G and CY in g.

By construction,
$$SA = e\,.\,AZ, \qquad (37)$$
$$SA' = e\,.\,A'Z \qquad (38)$$

(37) + (38), $\quad SA + SA' = e(AZ + A'Z)$,

i.e. $\quad (CS - CA) + (CS + CA') = e\,.\,AA'$,

i.e. $\quad 2CS = 2ae \qquad (\text{since } CA = CA' = a);$

$$\therefore\ CS = ae.$$

(38) − (37), $\quad SA' - SA = e(A'Z - AZ)$,

i.e. $\quad AA' = e[(A'C + CZ) - (CA - CZ)]$,

i.e. $\quad 2a = 2e\,.\,CZ$,

$$\therefore\ CZ = a/e.$$

Hence $\quad S \equiv (ae, 0),\ Z \equiv (a/e, 0).$

Now
$$PS^2 = (x - ae)^2 + y^2. \qquad (39)$$

From the diagram, $\quad PM = NZ = CN - CZ$
$$= x - a/e.$$

By definition,
$$SP = e\,.\,PM = ex - a, \qquad (40)$$
$$\therefore\ SP^2 = (ex - a)^2.$$

From (39) and (40),
$$(x - ae)^2 + y^2 = (ex - a)^2,$$

i.e.
$$x^2(1 - e^2) + y^2 = a^2(1 - e^2),$$

i.e.
$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1.$$

$(e^2 - 1)$ is positive, since $e^2 > 1$, therefore, using $b^2 = a^2(e^2 - 1)$ in this result, the standard equation of the hyperbola becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Note.—The notation used in this theorem will be used throughout.

40. Theorem.—To find the length of the semi-latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The latus rectum is the chord through S parallel to the directrix, i.e. parallel to the y-axis, and is therefore obtained by using $x = ae$ in the equation

$$x^2/a^2 - y^2/b^2 = 1,$$

giving
$$e^2 - y^2/b^2 = 1,$$

i.e.
$$\frac{y^2}{b^2} = e^2 - 1 = \frac{b^2}{a^2} \quad \text{(for extremities of latus rectum);}$$

$$\therefore\ y^2 = b^4/a^2, \quad \text{i.e. } y = \pm b^2/a.$$

Therefore the length of the semi-latus rectum is b^2/a.

41. Simple properties of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Using the equation in the form $y^2/b^2 = x^2/a^2 - 1$, it can be seen that, if y be real, then $|x| \geqslant a$, i.e. there is no portion of the curve between $x = \pm a$, and the curve must consist of two branches.

If $|x| > a$, there are two equal and opposite values of y for every value of x, and also for every value of y there are two equal and opposite values of x. Hence the curve is symmetrical about both axes.

From the symmetry of the curve K′L′ will be a second directrix with S′ as its corresponding focus.

As in the case of the ellipse, it can be shown that, if the point (x_1, y_1) lie on the curve, so also will the point $(-x_1, -y_1)$, and their join will pass through C, the points being equidistant from C. Hence C is the centre of the curve.

As $x \to \infty$, $y \to \pm\infty$, and therefore the hyperbola extends towards infinity in both directions.

The line AA′ is known as the *transverse axis*, and if B and B′ be points on CY such that $\text{CB} = \text{CB}' = b$, then BB′ is known as the *conjugate axis.*

42. Theorem.—To prove that, if $P \equiv (x_1, y_1)$ be any point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, then

$$SP = ex_1 - a, \quad S'P = ex_1 + a, \text{ and } S'P - SP = 2a.$$

Using the previous diagram, with $P \equiv (x_1, y_1)$,

$$PM = NZ = CN - CZ = x_1 - a/e,$$

$$\therefore\ SP = e\,.\,PM = ex_1 - a.$$

Also $$PM' = NZ' = CN + CZ' = x_1 + a/e.$$

$$\therefore\ S'P = e\,.\,PM' = ex_1 + a.$$

Hence $$S'P - SP = (ex_1 + a) - (ex_1 - a)$$
$$= 2a.$$

43. Comparison of the hyperbola and ellipse.

The standard equation $x^2/a^2 - y^2/b^2 = 1$ for the hyperbola is the same as that of the standard ellipse $x^2/a^2 + y^2/b^2 = 1$ with b^2 replaced by $-b^2$.

Thus certain properties of the hyperbola can be deduced from those of the ellipse by replacing b^2 by $-b^2$ in the corresponding result for the ellipse, wherever necessary.

The results for the hyperbola $x^2/a^2 - y^2/b^2 = 1$ thus obtained are as follows, using standard notation:

(*a*) The equations of the tangents of slope m are given by

$$y = mx \pm \sqrt{(a^2m^2 - b^2)}.$$

(*b*) The equation of the tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

(*c*) The equation of the polar of (h, k) is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1.$$

(*d*) The equation of the normal at (x_1, y_1) is

$$\frac{x - x_1}{x_1/a^2} + \frac{y - y_1}{y_1/b^2} = 0.$$

(*e*) The line $lx + my + n = 0$ touches the hyperbola if

$$a^2l^2 - b^2m^2 = n^2.$$

(*f*) The equation of the director circle (i.e. the locus of points of intersection of perpendicular tangents) is

$$x^2 + y^2 = a^2 - b^2.$$

It is a point circle if $a = b$, and imaginary if $a < b$.

(*g*) The normal bisects the angle between the focal radii SP and S'P, and the tangent bisects the external angle between the focal radii.

(*h*) $$\text{CN} \cdot \text{CT} = a^2.$$

(*i*) $$\text{CG} = e^2 x_1.$$

(*j*) $$\text{SR} \cdot \text{S'R'} = b^2.$$

(*k*) The locus of R and R′ is the auxiliary circle

$$x^2 + y^2 = a^2.$$

(*l*) The locus of mid-points of chords parallel to the line $y = mx$ is the line $y = m_1 x$, where $mm_1 = b^2/a^2$. These diameters $y = mx$ and $y = m_1 x$ are known as *conjugate diameters*.

44. Theorem.—To prove that, if one diameter of a hyperbola meet the curve in real points, then the conjugate diameter will meet it in imaginary points.

Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \qquad (41)$$

and let the equations of the conjugate diameters be

$$y = m_1 x, \qquad \qquad (42)$$

$$y = m_2 x. \qquad \qquad (43)$$

Where the line (42) meets the curve (41)

$$\frac{x^2}{a^2} - \frac{m_1{}^2 x^2}{b^2} = 1,$$

i.e. $$x^2 = a^2 b^2/(b^2 - a^2 m_1{}^2),$$

$$\therefore\ x = \pm \frac{ab}{\sqrt{(b^2 - a^2 m_1{}^2)}}. \qquad \qquad (44)$$

If the abscissæ given by (44) be real, then $b^2 > a^2 m_1{}^2$. Similarly, where the line (43) cuts the curve (41),

$$x = \pm \frac{ab}{\sqrt{(b^2 - a^2 m_2{}^2)}}. \qquad \qquad (45)$$

But $m_2 = b^2/(a^2m_1)$, since the diameters are conjugate, therefor from (45)

$$x = \pm ab \Big/ \sqrt{\left(b^2 - \frac{a^2b^4}{a^4m_1^2}\right)} = \pm \frac{a^2m_1}{\sqrt{(a^2m_1^2 - b^2)}}. \quad . \quad (46)$$

Now in (44) and (46) the quantities under the root sign are equa and opposite in sign. Hence if the values of x given in (44) are real those in (45) are imaginary, and vice versa.

Thus, *if one diameter cut the hyperbola in real points, its conjugat diameter must cut it in imaginary points.*

Note.—Since, for two conjugate diameters of slopes m_1 and m_2 $m_1m_2 = b^2/a^2$, it follows that the conjugate diameters are coinciden if $m_1 = m_2 = \pm b/a$.

Corollary.—Using this theorem, it can be seen that, if CP and CV be conjugate radii, P being a real point on the hyperbola and V the imaginary point, then

$$\mathrm{CP}^2 = \frac{a^2b^2}{b^2 - a^2m_1^2}(1 + m_1^2),$$

$$\mathrm{CV}^2 = \frac{a^4m_1^2}{a^2m_1^2 - b^2}(1 + m_2^2)$$

$$= \frac{a^4m_1^2 + a^4m_1^2m_2^2}{a^2m_1^2 - b^2} = -\frac{(a^4m_1^2 + b^4)}{b^2 - a^2m_1^2}.$$

$$\therefore\ \mathbf{CP^2 + CV^2} = \frac{a^2b^2 + a^2b^2m_1^2 - a^4m_1^2 - b^4}{b^2 - a^2m_1^2}$$

$$= \frac{a^2(b^2 - a^2m_1^2) - b^2(b^2 - a^2m_1^2)}{b^2 - a^2m_1^2}$$

$$= \boldsymbol{a^2 - b^2}.$$

45. Definition.—An asymptote to any curve is a line which cut the curve at two coincident points at infinity (i.e. a tangent to the curv at infinity).

Consider the quadratic equation

$$ax^2 + bx + c = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Using $x = 1/y$, it becomes on simplification

$$a + by + cy^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

The conditions that y shall have two coincident zero values in (48 are $a = 0$ and $b = 0$ simultaneously.

But when $y = 0$, the value of x is infinity. Hence the condition

that two roots of (47) shall be infinite are that $a = 0$ and $b = 0$ simultaneously.

46. Theorem.—To find the equations of the asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Let
$$y = mx + c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

be the equation of an asymptote of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Substituting from (49) in (50) for y, to obtain the abscissæ of the points of intersection of the line and hyperbola,

$$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1,$$

i.e.
$$x^2(b^2 - a^2m^2) - 2a^2mcx - a^2c^2 - a^2b^2 = 0. \quad . \quad . \quad (51)$$

If the line (49) be an asymptote to the curve (50), the roots of equation (51) in x must both be infinite, and the conditions for this are

$$b^2 - a^2m^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$a^2mc = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

From (52), $\quad m = \pm b/a.$

From (53), $\quad c = 0 \quad$ (since $m \neq 0$ and $a \neq 0$).

Therefore the equations of the asymptotes are

$$y = \pm(b/a)x,$$

which in the combined form is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

N.B.—The equations of the asymptotes in the combined form only differ from the equation of the hyperbola by a constant.

47. Definition.—The *conjugate hyperbola* to the hyperbola

$$x^2/a^2 - y^2/b^2 = 1$$

is the hyperbola whose equation is

$$y^2/b^2 - x^2/a^2 = 1,$$

and its two branches lie in the second and fourth quadrants.

48. Properties of conjugate hyperbolas.

(i) *Conjugate hyperbolas have the same asymptotes.*

(ii) *If two diameters be conjugate with respect to one hyperbola, they are also conjugate with respect to the conjugate hyperbola.*

(iii) *If two conjugate diameters cut one hyperbola in* P *and* P′ *and the conjugate hyperbola in* D *and* D′, *then* $CP^2 - CD^2 = a^2 - b^2$, *and also the parallelogram formed by the tangents at* P, P′, D, D′ *is of area* $4ab$.

49. Theorem.—To prove that, if 2α be the angle between the asymptotes of a hyperbola, and e be its eccentricity, then $\sec\alpha = e$.

Let the equation to the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Then the equations of the asymptotes are,

$$y = \pm(b/a)x,$$

from which it can be seen that the axes of co-ordinates bisect the angles between the asymptotes.

Hence $$\tan\alpha = b/a,$$

$$\therefore\ \sec^2\alpha = 1 + \tan^2\alpha = 1 + b^2/a^2$$
$$= 1 + (e^2 - 1) = e^2.$$

$$\therefore\ \sec\alpha = e \quad \text{(using } \alpha \text{ acute).}$$

50. Theorem.—To prove that the point $(x = a\sec\phi,\ y = b\tan\phi)$ lies on the hyperbola $x^2/a^2 - y^2/b^2 = 1$, for all values of ϕ.

Substituting $x = a\sec\phi$, $y = b\tan\phi$ in the equation

$$x^2/a^2 - y^2/b^2 = 1,$$

it becomes $\sec^2\phi - \tan^2\phi = 1$, which is true for all values of ϕ. Therefore $(a\sec\phi,\ b\tan\phi)$ lies on the hyperbola for all values of ϕ. Thus a parametric equation of the hyperbola is

$$x = a\sec\phi,\quad y = b\tan\phi.$$

51. Theorem.—To prove that the product of the perpendiculars from any point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ on to the asymptotes is constant and equal to $a^2b^2/(a^2 + b^2)$.

The hyperbola equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \ldots \ldots \quad (52)$$

and its asymptotes are given by

$$y = +(b/a)x, \quad \ldots \ldots \quad (53)$$

$$y = -(b/a)x. \quad \ldots \ldots \quad (54)$$

If $P \equiv (x_1, y_1)$ be any point on the hyperbola, the perpendicular (p_1) from P on the line (53) is given by

$$p_1 = \pm \frac{y_1 - (b/a)x_1}{\sqrt{(1 + b^2/a^2)}} = \pm \frac{ay_1 - bx_1}{\sqrt{(a^2 + b^2)}}.$$

Similarly, the perpendicular (p_2) from P on the line (54) is given by

$$p_2 = \pm \frac{ay_1 + bx_1}{\sqrt{(a^2 + b^2)}}.$$

$$\therefore\ p_1p_2 = \pm \frac{a^2y_1^2 - b^2x_1^2}{a^2 + b^2}$$

$$= \pm \left(\frac{y_1^2}{b^2} - \frac{x_1^2}{a^2}\right)\frac{a^2b^2}{a^2 + b^2}.$$

But $x_1^2/a^2 - y_1^2/b^2 = 1$, since (x_1, y_1) lies on curve (52), therefore, neglecting sign,

$$p_1p_2 = \frac{a^2b^2}{a^2 + b^2}.$$

52. Definition.—In the case of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $b = a$, the hyperbola is said to be "rectangular", since its asymptotes will be at right angles, and in this case its equation is $x^2 - y^2 = a^2$.

53. Theorem.—To find the equation of a rectangular hyperbola referred to its asymptotes as axes.

Let the equation of the hyperbola, referred to its own axes as axes of reference, be

$$x^2 - y^2 = a^2.$$

Let X and Y be the lengths of the perpendiculars from (x, y) on the curve $x^2 - y^2 = a^2$ to the asymptotes.

From the previous theorem (replacing b^2 by a^2)

$$XY = \frac{a^2 \times a^2}{a^2 + a^2}, \quad \text{i.e. } XY = a^2/2.$$

Since the perpendiculars on the asymptotes will be the co-ordinates of the point on the curve regarding the asymptotes as axes of reference, it follows that the equation to the rectangular hyperbola with its asymptotes as axes of reference is

$$\boldsymbol{xy = a^2/2.}$$

If the equation of the rectangular hyperbola be $xy = c^2$, it can readily be seen that, for all values of t, the point $(ct, c/t)$ must lie on the hyperbola. Hence a parametric equation of this hyperbola is

$$x = ct, \; y = c/t.$$

Using the hyperbola $xy = c^2$ and differentiating with respect to x,

$$x\frac{dy}{dx} + y = 0,$$

$$\therefore \frac{dy}{dx} = -\frac{y}{x}.$$

Therefore the slope of the tangent at (x_1, y_1) is $-y_1/x_1$, and the equation of the tangent at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{x_1}(x - x_1),$$

i.e. $$x_1y - x_1y_1 = -y_1x + x_1y_1,$$

i.e. $$y_1x + x_1y = 2x_1y_1.$$

But (x_1, y_1) lies on the hyperbola, $\therefore x_1y_1 = c^2$. Hence the equation of the tangent at (x_1, y_1) is

$$\boldsymbol{y_1x + x_1y = 2c^2.}$$

Example 6 (L.U.).—Find in symmetrical form the equation of the chord joining the two points $(ct, c/t)$ and $(ct_1, c/t_1)$ on the rectangular hyperbola $xy = c^2$

AB is a chord of a rectangular hyperbola which subtends a right angle at a fixed point P of the hyperbola. Show that AB must be parallel to a fixed direction, and that the circle on AB as diameter is one of a system of coaxal circles

The equation of the chord joining the points $(ct, c/t)$, and $(ct_1, c/t_1)$ is

$$\frac{x - ct}{ct - ct_1} = \frac{y - c/t}{c/t - c/t_1},$$

i.e.
$$\frac{x - ct}{t - t_1} = \frac{(ty - c)t_1}{t_1 - t},$$

i.e.
$$x - ct = -tt_1y + ct_1,$$

i.e.
$$x + tt_1y = c(t + t_1). \quad \ldots\ldots\ldots \quad \text{(i)}$$

Let A and B be the points $(ct, c/t)$ and $(ct_1, c/t_1)$ respectively, and $P \equiv (ct_2, c/t_2)$, where t_2 is fixed and t and t_1 are variables.

Slope of AP is
$$\left(\frac{c}{t} - \frac{c}{t_2}\right)\Big/(ct - ct_2) = -\frac{1}{tt_2}.$$

Slope of BP is
$$\left(\frac{c}{t_1} - \frac{c}{t_2}\right)\Big/(ct_1 - ct_2) = -\frac{1}{t_1t_2}.$$

Since $\angle APB = 90°$, the product of these slopes is -1,

i.e.
$$\frac{1}{tt_1t_2^2} = -1,$$

$$\therefore\ -\frac{1}{tt_1} = t_2^2.$$

But, from (i), the slope of the chord AB is $-1/tt_1$, therefore slope of chord AB is t_2^2, which is constant.

$\therefore$ AB is parallel to a fixed direction.

Using the formula $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$, for the equation of the circle on the join of (x_1, y_1) and (x_2, y_2) as diameter, the equation of a circle on AB as diameter is

$$(x - ct)(x - ct_1) + (y - c/t)(y - c/t_1) = 0,$$

i.e.
$$x^2 + y^2 - c(t + t_1)x - c\left(\frac{1}{t} + \frac{1}{t_1}\right)y + c^2\left(tt_1 + \frac{1}{tt_1}\right) = 0,$$

i.e.
$$x^2 + y^2 - c(t + t_1)x - \frac{c}{tt_1}(t + t_1)y + c^2\left(tt_1 + \frac{1}{tt_1}\right) = 0,$$

i.e.
$$x^2 + y^2 - \lambda x + \lambda t_2^2 y - c^2\left(\frac{1}{t_2^2} + t_2^2\right) = 0$$

(where λ is variable),

i.e.
$$x^2 + y^2 - \lambda(x - t_2^2y) - k = 0,$$

where k and t_2^2 are constant, and λ variable, which is the equation of a coaxal system of circles with radical axis $x = t_2^2y$.

Example 7 (L.U.).—Find the equation to the polar of the point (x_1, y_1) with respect to the rectangular hyperbola $xy = c^2$.

Prove that, if (x_1, y_1) lie on the curve $xy = a^2$, the previous polar touches the curve $a^2xy = c^4$.

Let the points of contact of the tangents (real, coincident, or imaginary) from (x_1, y_1) to the hyperbola $xy = c^2$ be (h_1, k_1), (h_2, k_2).

The equations of the tangents at these points are

$$xk_1 + yh_1 = 2c^2, \qquad \text{(i)}$$
$$xk_2 + yh_2 = 2c^2. \qquad \text{(ii)}$$

But the lines (i) and (ii) pass through (x_1, y_1),

$$\therefore\ x_1k_1 + y_1h_1 = 2c^2, \qquad \text{(iii)}$$
$$x_1k_2 + y_1h_2 = 2c^2. \qquad \text{(iv)}$$

The equations (iii) and (iv) are the conditions that the points (h_1, k_1), (h_2, k_2) shall lie on the line

$$x_1y + y_1x = 2c^2.$$

Thus, since the polar is the chord of contact, its equation will be

$$y_1x + x_1y = 2c^2. \qquad \text{(v)}$$

The tangent at (h, k) of the hyperbola

$$a^2xy = c^4 \qquad \text{(vi)}$$

is

$$a^2(kx + hy) = 2c^4. \qquad \text{(vii)}$$

Thus, if the line (v) touch the hyperbola (vi), the equations (v) and (vii) will be equivalent, therefore, comparing equations (v) and (vii),

$$\frac{a^2k}{y_1} = \frac{a^2h}{x_1} = \frac{2c^4}{2c^2},$$

$$\therefore\ x_1 = \frac{a^2h}{c^2}, \quad y_1 = \frac{a^2k}{c^2}.$$

Now (x_1, y_1) lies on the hyperbola $xy = a^2$,

$$\therefore\ x_1y_1 = a^2,$$

and it follows that $\quad a^4hk/c^4 = a^2$, i.e. $a^2hk = c^4$,

which is true, since (h, k) lies on the hyperbola (vi).

Hence the polar (v) touches the curve

$$a^2xy = c^4.$$

EXAMPLES ON CHAPTER X

All the following questions are taken from London University examination papers.

1. Find the condition that the lines $y = m_1x$, $y = m_2x$ may be conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Two conjugate diameters of this ellipse meet a directrix at P and Q. Through P a perpendicular is drawn to CQ, and through Q a perpendicular to CP. Prove that these perpendiculars meet in a fixed point.

2. Find the co-ordinates of N, the foot of the perpendicular from the centre of the ellipse $x^2/a^2 + y^2/b^2 = 1$, upon the tangent at the point P of the ellipse whose eccentric angle is θ.

If ϕ be the eccentric angle of the point of contact Q of the other tangent from N to the ellipse, show that $a^2 \tan \theta = b^2 \tan \frac{1}{2}(\theta + \phi)$, and verify that the normal at P passes through the extremity of the diameter through Q.

3. Find the condition that the lines $y = m_1 x$, $y = m_2 x$ shall be conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

If S and S′ be the foci of the above ellipse and PCP′, QCQ′ are two conjugate diameters, find the equation of the locus of the point of intersection of the perpendiculars through S and S′ to PCP′ and QCQ′ respectively.

4. A point moves so that the length of the tangent from it to a fixed circle is e times its distance from a fixed tangent to the circle. If $0 < e < 1$, prove that the locus of the point is an ellipse of eccentricity e, with one extremity of the major axis at the point of contact of the fixed tangent with the circle.

5. Show that the equation of the chord joining the points on the ellipse $x^2/a^2 + y^2/b^2 = 1$, whose eccentric angles are $(\alpha + \beta)$ and $(\alpha - \beta)$, is

$$bx \cos \alpha + ay \sin \alpha = ab \cos \beta.$$

If the chord passes through either focus of the ellipse, show that its length is

$$2a \sin^2 \beta.$$

6. Find the equations of the tangent and normal to the conic $ax^2 + by^2 = 1$ at the point (x_1, y_1).

The tangent at the point P of a central conic meets the axes of the conic at T and T′, and the normal at P meets the axes of the conic at N and N′. The perpendiculars to the axes at T and T′ meet at Q, and the perpendiculars at N and N′ meet at R. Show that the locus of R is a conic whose axes lie along the axes of the first conic, and that the tangent at R to its locus is the straight line RQ.

7. M is the mid-point of the chord AB of the ellipse $x^2/a^2 + y^2/b^2 = 1$. If the co-ordinates of M be (α, β), show that the equation of the line AB is

$$\frac{\alpha(x - \alpha)}{a^2} + \frac{\beta(y - \beta)}{b^2} = 0.$$

If the chord AB cut the axis of x at P and the axis of y at Q, and if $a^2/\mathrm{OP}^2 + b^2/\mathrm{OQ}^2$ is constant, where O is the origin, find the locus of the mid-point of AB.

8. Show that the equation of the straight line joining the points on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, whose eccentric angles are α and β, is

$$(x/a)(1 - \tan \tfrac{1}{2}\alpha \tan \tfrac{1}{2}\beta) + (y/b)(\tan \tfrac{1}{2}\alpha + \tan \tfrac{1}{2}\beta) = 1 + \tan \tfrac{1}{2}\alpha \tan \tfrac{1}{2}\beta.$$

A chord PQ of this ellipse passes through the point $(ka, 0)$ and P_1 is the reflection of P in the minor axis. Show that the equation of P_1Q is

$$(x/a)(1 - k \cos \alpha) + (ky/b) \sin \alpha = k - \cos \alpha,$$

where α is the eccentric angle of P.

9. Obtain in a simple form, symmetrical in α and β, the equation of the chord joining the points on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, whose eccentric angles are α and β.

A and A′ are the extremities of the major axis of the ellipse, B is either extremity of the minor axis, and P is any point on the ellipse. If X and Y be the points in which the diameter parallel to the tangent at P meets PA and PA′, show that the area of the triangle BXY is independent of the position of P.

10. Prove that the feet of the perpendiculars from the foci upon the tangents to an ellipse lie upon the auxiliary circle.

If SY be the perpendicular from the focus S, upon the tangent at P to an ellipse, show that the circumcircle of the triangle SPY touches the auxiliary circle.

11. If S and S′ are the foci of an ellipse, prove that the tangent at any point P on the ellipse makes equal angles with SP and S′P.

If TP and TQ are a pair of tangents from any point T to an ellipse, prove that the perpendicular distances of T from the lines SP, SQ, S′P, S′Q are all equal.

12. If P be any point on a conic, focus S, and the normal at P meets the axis through S in G, prove that SG = e . SP, where e is the eccentricity of the conic.

If PSP′ is a focal chord of a conic, and the normals at P and P′ meet parallels to the axis through S, through P′ and P respectively, in Q′ and Q, prove that QQ′ is parallel to PP′.

13. Prove that the tangents to a conic from an external point subtend equal or supplementary angles at a focus, and distinguish between the two cases.

A quadrilateral circumscribes an ellipse; prove that two opposite sides subtend supplementary angles at a focus.

14. A chord PQ of an ellipse cuts the directrix corresponding to the focus S in the point K. Show that SK bisects the angle PSQ externally.

Parallel lines are drawn in opposite senses from the foci S, S′ of an ellipse. The first, from S, meets the ellipse in P, and the other cuts the directrix corresponding to S′ in K′. K′P cuts the ellipse again in Q, and meets the directrix corresponding to S in K. Show that the difference between the angles SQS′, SPS′ is twice the angle PSK.

15. Find the equation of the chord joining the points $(ct_1, c/t_1)$, $(ct_2, c/t_2)$ of the rectangular hyperbola $xy = c^2$.

Three points P, Q, R are taken on this curve; show that the orthocentre H of the triangle PQR lies upon the curve and that the line joining the mid-points of QR and PH subtends a right angle at O.

16. Find the equation of the chord joining the points $(ct_1, c/t_1)$ and $(ct_2, c/t_2)$ on the rectangular hyperbola $xy = c^2$.

If the tangents at the points P and Q of this hyperbola intersect on the hyperbola $xy = 4c^2$, show that PQ touches the hyperbola $4xy = c^2$.

17. Find in a simple form, symmetrical in t_1 and t_2, the equation of the straight line joining the points $(ct_1, c/t_1)$ and $(ct_2, c/t_2)$ on the rectangular hyperbola $xy = c^2$.

The tangents at the points P and Q of the hyperbola meet one asymptote at

L and M, and meet the other at L′ and M′. Prove that PQ passes through the middle points of LM and L′M′.

18. Show that, if $t_1t_2t_3t_4 + 1 = 0$, the four points $(ct_1, c/t_1)$, $(ct_2, c/t_2)$, $(ct_3, c/t_3)$, $(ct_4, c/t_4)$ on the rectangular hyperbola $xy = c^2$ are the vertices and orthocentre of a triangle.

P, Q, R are the vertices of a triangle inscribed in a rectangular hyperbola. S is the orthocentre of the triangle. L, M, N are the feet of the perpendiculars drawn to one of the asymptotes from P, Q, R respectively. Show that the perpendiculars, drawn from L, M, N to QR, RP, PQ respectively, meet at the foot of the perpendicular drawn from S to the other asymptote.

19. Find, in symmetrical form, the equation of the chord joining the points $(ct, c/t)$, and $(ct_1, c/t_1)$ on the hyperbola $xy = c^2$.

A and B are two points on the hyperbola $xy = c^2$, whose centre is O. The mid-point of AB is M, and the mid-point of OM is N. If N be on the hyperbola, show that the chord AB touches the hyperbola $xy = 4c^2$.

20. Prove that the tangents from an external point to one branch of a hyperbola subtend equal angles at the corresponding focus.

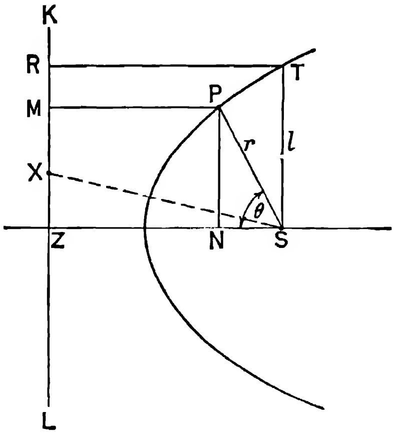
A tangent to a hyperbola meets the tangents at the vertices at P and Q. Prove that the circle on PQ as diameter passes through the foci, and deduce a geometrical construction for the asymptotes of a hyperbola given the foci and a vertex.

CHAPTER XI

The Polar and General Equation of a Conic

THE POLAR EQUATION OF A CONIC WITH THE FOCUS AS POLE

1. Let S be the focus of the conic and KL its directrix. SZ is the perpendicular from S on KL, and SZ is chosen as the initial line with S as the pole of co-ordinates. $P \equiv (r, \theta)$ is any point on the conic, and ST the semi-latus rectum of length l.



The eccentricity of the conic is e, and TR is the perpendicular from T on KL. PN is the ordinate of P, and PM the perpendicular from P on KL.

From the definition of a conic,

$$ST = e \,.\, TR, \quad \therefore \; TR = l/e.$$

From the diagram, $$SZ = TR = l/e.$$

Now $$SN = r \cos\theta, \quad \text{and} \quad SP = ePM,$$

$$\therefore \; PM = NZ = r/e.$$

Also, $$SZ = NZ + NS,$$

$$\therefore \; l/e = r/e + r\cos\theta,$$

i.e. the polar equation of the conic is

$$\frac{l}{r} = 1 + e \cos \theta.$$

2. Theorem.—To find the polar equation of the directrix of the conic $l/r = 1 + e \cos \theta$.

With the previous notation and diagram, let $X \equiv (r, \theta)$ be any point on the directrix KL.

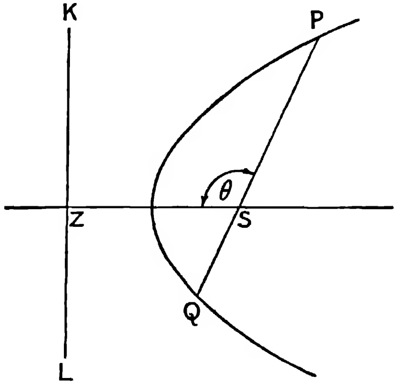
Then $$SZ = SX \cos \theta$$
$$= r \cos \theta.$$

But $$SZ = l/e, \quad \therefore \; l/e = r \cos \theta,$$

i.e. the required equation is

$$\frac{l}{r} = e \cos \theta.$$

3. Theorem.—To prove that, if r_1 and r_2 be the segments into which a focal chord of the conic be divided by the focus, then $\dfrac{1}{r_1} + \dfrac{1}{r_2} = \dfrac{2}{l}$.



Let the polar equation of the conic, with usual notation, be

$$\frac{l}{r} = 1 + e \cos \theta.$$

PQ is the focal chord with $P \equiv (r_1, \theta)$.

$$\therefore \; Q \equiv (r_2, 180° + \theta).$$

Since P and Q lie on the conic,

$$\frac{l}{r_1} = 1 + e\cos\theta, \quad \ldots\ldots \quad (1)$$

$$\frac{l}{r_2} = 1 + e\cos(180^\circ + \theta)$$

$$= 1 - e\cos\theta. \quad \ldots\ldots \quad (2)$$

(1) + (2),

$$\frac{l}{r_1} + \frac{l}{r_2} = 2,$$

$$\therefore \frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{l}.$$

4. Theorem.—To find the equation of the chord joining two points on the conic $l/r = 1 + e\cos\theta$.

If A, B be two arbitrary constants, the equation

$$\frac{l}{r} = A\cos\theta + B\cos(\theta - \alpha) \quad \ldots\ldots \quad (3)$$

will represent the equation of the straight line passing through any two points.

Let the vectorial angles of the two points on the curve be $(\alpha + \beta)$ and $(\alpha - \beta)$, and let the vectors be r_1 and r_2.

Since the points lie on the line (3),

$$\frac{l}{r_1} = A\cos(\alpha + \beta) + B\cos\beta, \quad \ldots\ldots \quad (4)$$

$$\frac{l}{r_2} = A\cos(\alpha - \beta) + B\cos\beta. \quad \ldots\ldots \quad (5)$$

Since the points also lie on the conic,

$$\frac{l}{r_1} = 1 + e\cos(\alpha + \beta), \quad \ldots\ldots \quad (6)$$

$$\frac{l}{r_2} = 1 + e\cos(\alpha - \beta). \quad \ldots\ldots \quad (7)$$

From (4) and (6),

$$A\cos(\alpha + \beta) + B\cos\beta = 1 + e\cos(\alpha + \beta). \quad . \quad (8)$$

From (5) and (7),

$$A\cos(\alpha-\beta)+B\cos\beta=1+e\cos(\alpha-\beta). \quad . \quad (9)$$

(8) — (9) gives

$$A[\cos(\alpha+\beta)-\cos(\alpha-\beta)]=e[\cos(\alpha+\beta)-\cos(\alpha-\beta)].$$

$$\therefore\ A=e.$$

Hence $$B\cos\beta=1, \quad \therefore\ B=\sec\beta.$$

Thus the required equation of the chord is

$$\frac{l}{r}=e\cos\theta+\sec\beta\cos(\theta-\alpha).$$

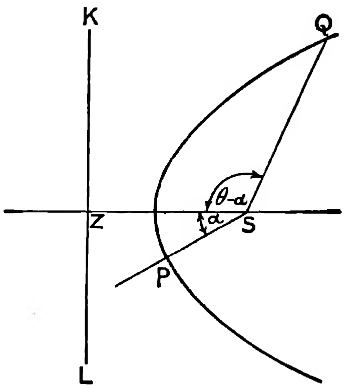
When $\beta=0$, the chord becomes the tangent at the point (r, α) and its equation is

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\alpha).$$

Example 1 (L.U.).—Prove that the polar equation of a conic with its focus at the origin may be put in the form $l/r=1+e\cos(\theta-\alpha)$.

An ellipse has a given focus S. Given the eccentricity e and the fact that it passes through a fixed point P, where $SP=p$, also taking S as the pole of co-ordinates and SP as the initial line, show that the polar equation of the locus of the second focus of the ellipse may be put in the form

$$(1-e^2)r=2ep(1-e\cos\theta).$$



The proof of the first part of the question is the same as that for proving the polar equation to be $l/r=1+e\cos\theta$, with SZ as the initial line, but now SP is the initial line making an angle α with SZ, and $Q\equiv(r, \theta)$ is any point on the conic.

In any one case, KL is the directrix and SZ is perpendicular to KL. The semi-latus rectum is l and SP makes an angle α with SZ. (*N.B.* α and l are variables.)

The equation of the conic will then be

$$\frac{l}{r} = 1 + e\cos(\theta - \alpha), \quad \dots\dots\dots \quad \text{(i)}$$

Since $P \equiv (p, 0)$ lies on the curve,

$$\frac{l}{p} = 1 + e\cos\alpha. \quad \dots\dots\dots\dots \quad \text{(ii)}$$

Using the properties of the ellipse found in Cartesian co-ordinates, where $2a$ is the major axis and S′ the second focus,

$$SS' = 2ae = r, \text{ if } S' \equiv (r, \theta).$$

Now
$$l = \frac{b^2}{a} = \frac{a^2(1 - e^2)}{a} = a(1 - e^2)$$

$$= \frac{r}{2e}(1 - e^2). \quad \dots\dots\dots\dots \quad \text{(iii)}$$

Also
$$\theta = 180^\circ + \alpha, \; \therefore \alpha = \theta - 180^\circ, \quad \dots\dots \quad \text{(iv)}$$

Substituting from (iii) and (iv) in (ii),

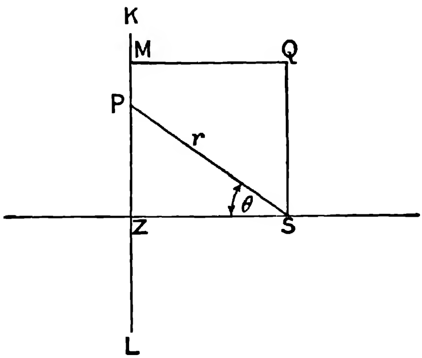
$$\frac{r(1 - e^2)}{2ep} = 1 + e\cos(\theta - 180^\circ)$$

$$= 1 - e\cos\theta.$$

Thus required equation of locus of second focus is

$$r(1 - e^2) = 2ep(1 - e\cos\theta).$$

Example 2 (L.U.).—Find the equation of the directrix corresponding to the pole as focus of the conic $l/r = 1 + e\cos\theta$.



Show in a figure this focus and directrix, and the circle $er = 2l\sin\theta$, and find the co-ordinates of the extremities of a chord of the conic which passes through the pole and is cut harmonically by the circle.

For any point $P \equiv (r, \theta)$ on the directrix KL, $SZ/SP = \cos\theta$, where S is the focus, and $SZ = l/e$ is the perpendicular from S on KL,

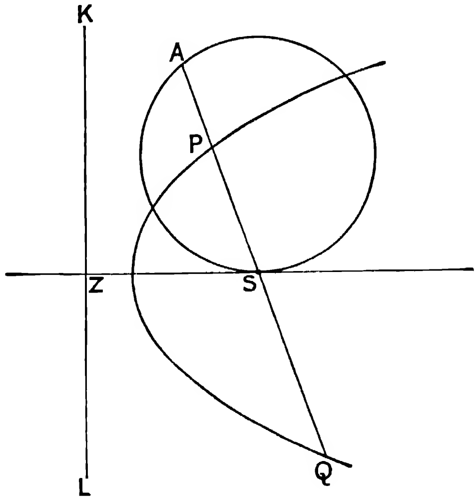
i.e. $$l/(er) = \cos\theta.$$

Hence the equation of the directrix, as shown previously, is

$$l/r = e\cos\theta.$$

N.B.—If $SQ = l$ be the semi-latus rectum, then $SQ = e \,.\, QM$, where $QM =$ perpendicular from Q on KL, i.e. $l = e \,.\, SZ$, $\therefore\ SZ = l/e$.

Let the extremities of the chord be (r_1, θ), $(r_2, \theta + \pi)$, with S the focus (i.e. the pole of co-ordinates) and A the point (r, θ) on the circle $er = 2l\sin\theta$, in which QP produced cuts the circle.



The condition that APSQ shall form a harmonic range is

$$\frac{QS \,.\, PA}{QA \,.\, PS} = -1 \quad \text{(definition of \textit{harmonic range})},$$

i.e. considering only magnitudes and not directions,

$$QS \,.\, PA = QA \,.\, SP,$$

i.e. $$r_2(r - r_1) = (r + r_2)r_1. \qquad \text{(i)}$$

Since P and Q lie on the conic $l/r = 1 + e\cos\theta$,

$$r_1 = \frac{l}{1 + e\cos\theta}, \qquad \text{(ii)}$$

$$r_2 = \frac{l}{1 - e\cos\theta}. \qquad \text{(iii)}$$

Also, A lies on the circle $er = 2l \sin \theta$,

$$\therefore\ r = \frac{2l \sin \theta}{e}. \qquad \ldots \ldots \ldots \text{(iv)}$$

Using (ii), (iii), (iv), in (i),

$$\frac{l}{1 - e \cos \theta}\left(\frac{2l \sin \theta}{e} - \frac{l}{1 + e \cos \theta}\right) = \left(\frac{2l \sin \theta}{e} + \frac{l}{1 - e \cos \theta}\right)\frac{l}{1 + e \cos \theta},$$

i.e. $$2 \sin \theta(1 + e \cos \theta) - e = 2 \sin \theta(1 - e \cos \theta) + e,$$

i.e. $$4e \sin \theta \cos \theta = 2e.$$

$$\therefore\ \sin 2\theta = 1.$$

$$\therefore\ \theta = \tfrac{1}{4}\pi \qquad \text{(using } \theta \text{ acute).}$$

With $\theta = \frac{1}{4}\pi$,

from (ii), $$r_1 = l\sqrt{2}/(\sqrt{2} + e);$$

from (iii), $$r_2 = l\sqrt{2}/(\sqrt{2} - e).$$

Thus the required points are

$$\left(\frac{l\sqrt{2}}{\sqrt{2} + e}, \frac{\pi}{4}\right); \quad \left(\frac{l\sqrt{2}}{\sqrt{2} - e}, \frac{5\pi}{4}\right).$$

Example 3 (L.U.).—If the vectorial angles of the points A and B on the conic $1 + e \cos \theta = l/r$ are $(\alpha - \beta)$ and $(\alpha + \beta)$, show that the equation of the chord AB is

$$\sec \beta \cos(\theta - \alpha) + e \cos \theta = l/r.$$

Find the points of intersection of the straight line $2 \cos \theta + \tan \alpha \sin \theta = l/r$ and the conic $1 + \cos \theta = l/r$, and show that the tangents to the conic at these points cut each other at the angle α.

The first part of the question is proved in Section 4.

$$2 \cos \theta + \tan \alpha \sin \theta = \frac{l}{r}, \qquad \ldots \ldots \ldots \text{(i)}$$

$$1 + \cos \theta = \frac{l}{r}. \qquad \ldots \ldots \ldots \text{(ii)}$$

Where the line (i) and the conic (ii) intersect, by taking (i) — (ii),

$$\cos \theta - 1 + \tan \alpha \sin \theta = 0,$$

i.e. $$\tan \alpha \sin \theta = 1 - \cos \theta,$$

i.e. $$2 \tan \alpha \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta = 2 \sin^2 \tfrac{1}{2}\theta.$$

$$\therefore\ \tan \tfrac{1}{2}\theta = \tan \alpha, \text{ or } \sin \tfrac{1}{2}\theta = 0.$$

$$\therefore\ \tfrac{1}{2}\theta = \alpha, \text{ or } \tfrac{1}{2}\theta = 0,$$

i.e. $$\theta = 2\alpha \text{ or } 0.$$

From (ii), $$r = l/(1 + \cos \theta)$$

$$= l/(1 + \cos 2\alpha) \qquad \text{(when } \theta = 2\alpha\text{);}$$

and $$r = l/2 \qquad \text{(when } \theta = 0\text{).}$$

Therefore the required points are

$$\left(\frac{l}{2}, 0\right) \text{ and } \left(\frac{l}{1+\cos 2\alpha}, 2\alpha\right).$$

From the first part of the question, the tangent at the point where the vectorial angle is 2α (using $e = 1$ and $\beta = 0$) is

$$\frac{l}{r} = \cos(\theta - 2\alpha) + \cos\theta. \quad \ldots\ldots \quad \text{(iii)}$$

Rewriting the equation of the line (iii) as

$$l = r(\cos\theta\cos 2\alpha + \sin\theta\sin 2\alpha) + r\cos\theta$$
$$= x\cos 2\alpha + y\sin 2\alpha + x$$

(since $x = r\cos\theta$, $y = r\sin\theta$),

its slope is
$$-\frac{(1+\cos 2\alpha)}{\sin 2\alpha} = -\frac{2\cos^2\alpha}{2\sin\alpha\cos\alpha} = -\cot\alpha,$$

i.e. the tangent at this point makes an angle α with a perpendicular to the initial line. But the tangent where $\alpha = 0$ is perpendicular to the initial line, therefore the tangents at the points of intersection are at an angle α.

The General Equation of a Conic

5. Theorem.—To show that the general equation of the second degree in x and y, namely

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

represents a conic, i.e. a pair of straight lines, a circle, a parabola, an ellipse, or a hyperbola.

Let the axes of co-ordinates OX, OY be turned through an angle θ, the new axes being OX′, OY′, and let any point whose co-ordinates were (x, y) with respect to the original axes have co-ordinates (X, Y) with respect to the new axes.

Then,
$$x = X\cos\theta - Y\sin\theta,$$
$$y = X\sin\theta + Y\cos\theta$$
(see p. 244).

Substituting these values in the general equation of the second degree, the equation with respect to the new axes becomes

$$a(X\cos\theta - Y\sin\theta)^2 + 2h(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta)$$
$$+ b(X\sin\theta + Y\cos\theta)^2 + 2g(X\cos\theta - Y\sin\theta)$$
$$+ 2f(X\sin\theta + Y\cos\theta) + c = 0,$$

i.e. $X^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)$

$$+ 2XY(-a\sin\theta\cos\theta + h\cos^2\theta - h\sin^2\theta + b\sin\theta\cos\theta)$$
$$+ Y^2(a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta) + 2X(g\cos\theta + f\sin\theta)$$
$$+ 2Y(f\cos\theta + g\sin\theta) + c = 0.$$

Now choose θ so that the coefficient of XY vanishes,

i.e. $$h(\cos^2\theta - \sin^2\theta) = (a - b)\sin\theta\cos\theta,$$

i.e. $$2h\cos 2\theta = (a - b)\sin 2\theta,$$

i.e. $$\tan 2\theta = \frac{2h}{a - b},$$

giving a unique value of θ.

The equation of the curve can now be written

$$AX^2 + BY^2 + 2GX + 2FY + C = 0, \quad . \quad . \quad (10)$$

where A, B, C, etc., are constants.

This can be written

$$A\left(X + \frac{G}{A}\right)^2 + B\left(Y + \frac{F}{B}\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - C,$$

and transferring the origin to the point $\left(\frac{-G}{A}, \frac{-F}{B}\right)$, the equation of the curve becomes

$$AX^2 + BY^2 = D. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If $D = 0$, the equation clearly represents a *pair of straight lines.*

If $D \neq 0$, and $A = B$, the equation represents a *circle.*

The equation (11) can also be written

$$\frac{X^2}{D/A} + \frac{Y^2}{D/B} = 1.$$

From this it is clear that, if $D \neq 0$, and $A \neq B$, the equation represents a *real ellipse* if D/A and D/B be both positive, and an *imaginary ellipse* if D/A and D/B be both negative. Also if D/A and D/B are of opposite signs, the equation represents a *hyperbola.*

In this discussion it has been assumed that neither A nor B is zero.

Consider now $A = 0$. (A and B cannot be simultaneously zero or there would be no conic.)

When $A = 0$, the equation (10) becomes

$$BY^2 + 2GX + 2FY + C = 0,$$

i.e.
$$B\left(Y + \frac{F}{B}\right)^2 = \frac{F^2}{B} - C - 2GX.$$

If G is also zero, it can be seen that the equation represents a pair of *parallel lines*.

If $G \neq 0$, the equation can be written

$$B\left(Y + \frac{F}{B}\right)^2 = -2G\left\{X - \left(\frac{F^2}{2BG} - \frac{C}{2G}\right)\right\},$$

which is the equation of a *parabola* with its vertex at the point

$$\left\{\left(\frac{F^2}{2BG} - \frac{C}{2G}\right), \frac{-F}{B}\right\}.$$

Thus the general equation of the second degree in x and y represents a conic section in all cases.

6. Theorem.—To find the co-ordinates of the centre of the conic represented by the equation $f(x, y) = 0$, where

$$f(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c.$$

The centre of a conic has been defined as the point that bisects all chords of the conic passing through it. (The parabola has no centre.) Thus if this point be chosen as the origin of co-ordinates it follows that, if (x_1, y_1) lie on the curve, so also will the point $(-x_1, -y_1)$. Hence it must follow that, if the origin of co-ordinates be the centre of the conic, the terms in x and y must vanish.

Let (x_1, y_1) be the centre of the given conic. Transferring the origin to this point, the equation of the curve becomes

$$a(x + x_1)^2 + b(y + y_1)^2 + 2h(x + x_1)(y + y_1) + 2g(x + x_1) + 2f(y + y_1) + c = 0,$$

i.e.
$$ax^2 + 2hxy + by^2 + 2x(ax_1 + hy_1 + g) + 2y(hx_1 + by_1 + f) + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0. \quad . \quad (12)$$

Since the centre of the conic is the origin of co-ordinates, it follows that

$$ax_1 + hy_1 + g = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$hx_1 + by_1 + f = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\left[\text{i.e. } \left(\frac{\partial f}{\partial x}\right)_1 = 0 \text{ and } \left(\frac{\partial f}{\partial y}\right)_1 = 0 \text{ respectively}\right].$$

Solving (13) and (14) for x_1 and y_1,

$$\frac{x_1}{hf - bg} = \frac{-y_1}{af - hg} = \frac{1}{ab - h^2},$$

i.e.
$$x_1 = \frac{hf - bg}{ab - h^2}, \quad y_1 = \frac{hg - af}{ab - h^2}.$$

Hence the centre of the conic is the point

$$\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right).$$

If $ab = h^2$, the centre of the conic is at infinity, and the conic is a *parabola* (provided $hf \neq bg$).

If in addition to $ab = h^2$, $hf = bg$, then

$$a/h = h/b = g/f,$$

and the equations (13) and (14) are identical, i.e. the centre is some point on a given straight line, and in this case the equation represents a *pair of parallel straight lines.* Multiplying (13) by x_1 and (14) by y_1, and adding the results,

$$ax_1^2 + 2hx_1y_1 + by_1^2 + gx_1 + fy_1 = 0.$$

Adding $gx_1 + fy_1 + c$ to each side of this,

$$\begin{aligned} ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c &= gx_1 + fy_1 + c \\ &= \frac{g(hf - bg)}{ab - h^2} + \frac{f(hg - af)}{ab - h^2} + c \\ &= \frac{2fgh + abc - af^2 - bg^2 - ch^2}{ab - h^2} \end{aligned}$$

$$ab - h^2, \quad \text{where } \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Using this result, and results (13) and (14) in (12), the equation of the conic referred to its centre as origin is

$$\boldsymbol{ax^2 + 2hxy + by^2 + \frac{}{ab - h^2} = 0.}$$

From this equation it can be seen that the condition that the general equation of the second degree shall represent a pair of straight lines is

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

The quantity Δ is known as the *discriminant* of the original equation $f(x, y) = 0$.

7. Theorem.—To find the position and magnitude of the axes of a central conic (i.e. a conic having a finite centre).

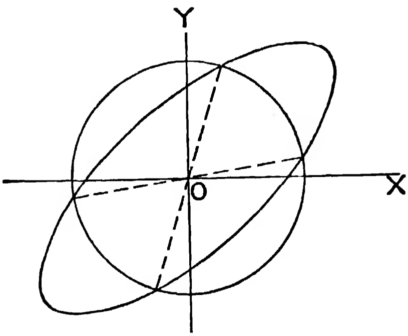
Taking the centre of the conic as origin, its equation can be taken as

$$ax^2 + 2hxy + by^2 = 1. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The equation of a circle centre O, radius r will be

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1. \qquad (16)$$

The circle given by equation (16) will cut the conic given by equation (15) in four points, symmetrically placed with respect to O, and hence lying on a pair of straight lines through the origin.



As this pair of straight lines move into coincidence, by changing the value of r, they tend to coincide with the axes of the conic given by equation (15), and when they coincide with the axes of the conic, the values of r will give the lengths of the axes of the conic.

(15) — (16) gives

$$x^2\left(a - \frac{1}{r^2}\right) + 2hxy + y^2\left(b - \frac{1}{r^2}\right) = 0, \quad . \quad . \quad (17)$$

which is the equation of a pair of straight lines through the origin, and the points of intersection of the circle and conic. These lines are coincident when the roots of (17) are coincident, i.e. when

$$h^2 = \left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Now the equation (18) can be transformed into a quadratic in r^2, which will therefore give the two values of the axes of the conic. Multiplying (17) by $(b - 1/r^2)$, it becomes

$$x^2\left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right) + 2h\left(b - \frac{1}{r^2}\right)xy + y^2\left(b - \frac{1}{r^2}\right)^2 = 0.$$

Using (18) in this, it becomes

$$h^2x^2 + 2h\left(b - \frac{1}{r^2}\right)xy + \left(b - \frac{1}{r^2}\right)^2 y^2 = 0,$$

i.e.

$$\left\{hx + \left(b - \frac{1}{r^2}\right)y\right\}^2 = 0.$$

Hence, with the appropriate values for r^2, the equations of the axes are given by

$$hx + \left(b - \frac{1}{r^2}\right)y = 0.$$

8. Theorem.—To find the equations of the asymptotes of the general conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

When dealing with the hyperbola, which is the only conic having asymptotes, it was seen that the equations of the asymptotes only differed from the equation of the curve by a constant. Thus it can be assumed that the asymptotes in this case are given by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda = 0, \quad . \quad (19)$$

where λ is a constant.

The condition that equation (19) shall represent a pair of straight lines is

$$ab(c + \lambda) + 2fgh - af^2 - bg^2 - (c + \lambda)h^2 = 0,$$

from which

$$\begin{aligned}\lambda &= -(abc + 2fgh - af^2 - bg^2 - ch^2)/(ab - h^2) \\ &= -\Delta/(ab - h^2).\end{aligned}$$

Therefore the equation of the asymptotes is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{\Delta}{ab - h^2} = 0.$$

9. Theorem.—To find the axis and latus rectum of a parabola.

When the axis of the parabola is chosen as the x-axis, and its vertex as origin, the directrix being parallel to the y-axis, the equation of the parabola is $y^2 = 4ax$, where $4a$ is the latus rectum.

Hence the equation of a parabola whose tangent at the vertex is $lx + my + n = 0$, and whose axis is $\alpha x + \beta y + \gamma = 0$, will be

$$\left(\frac{\alpha x + \beta y + \gamma}{\sqrt{(\alpha^2 + \beta^2)}}\right)^2 = 4p\,\frac{lx + my + n}{\sqrt{(l^2 + m^2)}},$$

where $4p$ is the latus rectum.

Thus, if the given equation of the parabola can be expressed in this form (remembering that the axis of the parabola is perpendicular to the tangent at the vertex), it is possible to read off the equations of the axis and tangent at the vertex to the parabola, and also to write down the length of the latus rectum.

Example 4 (L.U.).—(*a*) Prove that the conic $13(x^2 + y^2) = (3x + 2y - b)^2$ is a parabola, and find the length of its latus rectum.

(*b*) Find the equation of the hyperbola whose asymptotes are parallel to the lines $x + y = 0$ and $3x + 2y = 0$, which touches the x-axis at the origin, and passes through the point $(1, -2)$.

(*a*) The given equation can be written

$$4x^2 - 12xy + 9y^2 + 6bx + 4by - b^2 = 0.$$

Now the square of half the coefficient of xy is 36 and the product of the coefficients of x^2 and y^2 is 36, and therefore the equation represents a parabola. ($h^2 = ab$ with usual notation.)

The equation can also be written

$$\left(\frac{2x - 3y + c}{\sqrt{13}}\right)^2 = \tfrac{1}{13}(-6bx - 4by + b^2 + 4cx - 6cy + c^2),$$

where c is a constant to be chosen later,

i.e. $$\left(\frac{2x - 3y + c}{\sqrt{13}}\right)^2 = \tfrac{1}{13}\{x(4c - 6b) - y(6c + 4b) + b^2 + c^2\}.$$

Now choose c so that the lines represented by $2x - 3y + c = 0$ and $x(4c - 6b) - y(6c + 4b) + b^2 + c^2 = 0$ are perpendicular.

Then $$\frac{2}{3}\left(\frac{4c - 6b}{6c + 4b}\right) = -1,$$

i.e. $$8c - 12b = -18c - 12b,$$

$$\therefore\ c = 0.$$

Thus the equations of the parabola can be written

$$\left(\frac{2x-3y}{\sqrt{13}}\right)^2 = \frac{2\sqrt{13}\,.\,b}{13}\left[\frac{b-6x-4y}{2\sqrt{13}}\right].$$

From this equation it can be seen that the length of the latus rectum is $\dfrac{2\sqrt{13}\,.\,b}{13}$.

(*b*) The equation of the hyperbola whose asymptotes are parallel to the lines $x+y=0$ and $3x+2y=0$ will be $(x+y+\lambda_1)(3x+2y+\lambda_2)+\lambda_3=0$, where λ_1, λ_2, λ_3 are constants.

When $y=0$, i.e. where it cuts OX,

$$(x+\lambda_1)(3x+\lambda_2)+\lambda_3=0,$$

i.e.
$$3x^2+x(3\lambda_1+\lambda_2)+\lambda_1\lambda_2+\lambda_3=0.$$

Since the hyperbola touches OX at O, it follows that

$$3\lambda_1+\lambda_2=0, \quad \ldots\ldots\ldots \quad \text{(i)}$$

and
$$\lambda_1\lambda_2+\lambda_3=0. \quad \ldots\ldots\ldots \quad \text{(ii)}$$

Since the curve passes through (1, −2),

$$(\lambda_1-1)(\lambda_2-1)+\lambda_3=0. \quad \ldots\ldots\ldots \quad \text{(iii)}$$

Substituting from (ii) in (iii) for λ_3,

$$\lambda_1\lambda_2-\lambda_1-\lambda_2+1-\lambda_1\lambda_2=0,$$

i.e.
$$\lambda_1+\lambda_2=1. \quad \ldots\ldots\ldots \quad \text{(iv)}$$

From (i) and (iv), $\lambda_1=-\frac{1}{2},\ \lambda_2=\frac{3}{2}$,

and from (ii), $\lambda_3=\frac{3}{4}$.

Therefore the equation of the hyperbola is

$$(x+y-\tfrac{1}{2})(3x+2y+\tfrac{3}{2})+\tfrac{3}{4}=0,$$

i.e.
$$(2x+2y-1)(6x+4y+3)+3=0,$$

i.e.
$$12x^2+20xy+8y^2+2y=0,$$

i.e.
$$6x^2+10xy+4y^2+y=0.$$

Example 5 (L.U.).—Show that the conic whose equation is

$$a^2x^2+2hxy+b^2y^2-2acx-2bcy+c^2=0$$

touches both co-ordinate axes.

Find the co-ordinates of its centre and the equation of its director circle (i.e. the locus of the intersections of perpendicular tangents to the conic), and show that, for all values of h, the director circle touches the line $bx+ay=0$.

$$a^2x^2+2hxy+b^2y^2-2acx-2bcy+c^2=0. \quad \ldots\ldots \quad \text{(i)}$$

Where $y=0$, i.e. where the conic cuts OX,

$$a^2x^2-2acx+c^2=0,$$

i.e. $x=c/a$ twice, therefore the conic touches OX.

Where $x = 0$, i.e. where the conic cuts OY,

$$b^2y^2 - 2bcy + c^2 = 0,$$

i.e. $y = c/b$ twice, therefore the conic touches OY, and therefore the conic touches both co-ordinate axes.

For the centre of the conic $f(x, y) = 0$,

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0,$$

i.e., for the conic (i), $\quad 2a^2x + 2hy - 2ac = 0,$

i.e. $\quad a^2x + hy - ac = 0, \quad \dots\dots$ (ii)

and $\quad 2hx + 2b^2y - 2bc = 0,$

i.e. $\quad hx + b^2y - bc = 0. \quad \dots\dots$ (iii)

Solving (ii) and (iii) by determinants,

$$\frac{x}{-hbc + ab^2c} = \frac{-y}{-a^2bc + hac} = \frac{1}{a^2b^2 - h^2}.$$

$$\therefore\ x = \frac{bc(ab - h)}{a^2b^2 - h^2} = \frac{bc}{ab + h},$$

$$y = \frac{ac(ab - h)}{a^2b^2 - h^2} = \frac{ac}{ab + h}.$$

Therefore the co-ordinates of the centre are

$$\left(\frac{bc}{ab + h}, \frac{ac}{ab + h}\right).$$

The centre of the director circle is the centre of the conic,

$$\left(\frac{bc}{ab + h}, \frac{ac}{ab + h}\right),$$

and since the conic touches the axes OX, OY, these axes are perpendicular tangents, and O is a point on the director circle. Hence the radius of the director circle is the distance from O to the point

$$\left(\frac{bc}{ab + h}, \frac{ac}{ab + h}\right),$$

i.e. $$\sqrt{\left\{\frac{b^2c^2}{(ab + h)^2} + \frac{a^2c^2}{(ab + h)^2}\right\}} = \frac{c}{ab + h}\sqrt{(a^2 + b^2)}.$$

Therefore the equation of the director circle is

$$\left(x - \frac{bc}{ab + h}\right)^2 + \left(y - \frac{ac}{ab + h}\right)^2 = \frac{b^2c^2}{(ab + h)^2} + \frac{a^2c^2}{(ab + h)^2},$$

i.e. $$x^2 + y^2 - \frac{2bc}{ab + h}x - \frac{2ac}{ab + h}y = 0,$$

i.e. $$(ab + h)(x^2 + y^2) - 2bcx - 2acy = 0.$$

The perpendicular from the point

$$\left(\frac{bc}{ab+h}, \frac{ac}{ab+h}\right)$$

on the line $bx + ay = 0$ is

$$\left(\frac{b^2c}{ab+h} + \frac{a^2c}{ab+h}\right)\Big/\sqrt{(b^2+a^2)} = \frac{c}{ab+h}\cdot\frac{a^2+b^2}{\sqrt{(a^2+b^2)}} = \frac{c}{ab+h}\cdot\sqrt{(a^2+b^2)}$$

$$= \text{radius of the director circle.}$$

Therefore the director circle touches the line $bx + ay = 0$.

Example 6 (L.U.).—Find the separate equations of the asymptotes of the hyperbola

$$3x^2 + 2xy - 8y^2 - 8x + 14y + 2 = 0.$$

Find the equation of the hyperbola whose asymptotes are parallel to those of the given hyperbola, which touches the line $y = x$ at the point (1, 1), and which passes through the point (3, 2). Also find the separate equations of its asymptotes.

Working from first principles, let

$$y = mx + c \quad \dots\dots\dots \quad \text{(i)}$$

be an asymptote of the given hyperbola.

Where the line (i) cuts the given hyperbola, by substitution for y,

$$3x^2 + 2x(mx + c) - 8(mx + c)^2 - 8x + 14(mx + c) + 2 = 0,$$

i.e. $$x^2(3 + 2m - 8m^2) + 2x(c - 8mc - 4 + 7m) - 8c^2 + 14c + 2 = 0.$$

Since line (i) is an asymptote, the coefficients of x^2 and x must vanish simultaneously, i.e.

$$3 + 2m - 8m^2 = 0, \quad \dots\dots\dots \quad \text{(ii)}$$

$$c - 8mc - 4 + 7m = 0. \quad \dots\dots\dots \quad \text{(iii)}$$

From (ii), $(3 - 4m)(1 + 2m) = 0, \ \therefore m = \frac{3}{4}$ or $-\frac{1}{2}$.

Using $m = \frac{3}{4}$ in (iii), $c - 6c - 4 + \frac{21}{4} = 0,$

i.e. $\frac{5}{4} = 5c, \ \therefore c = \frac{1}{4}.$

Using $m = -\frac{1}{2}$ in (iii), $c + 4c - 4 - \frac{7}{2} = 0,$

$\therefore 5c = \frac{15}{2}, \ \therefore c = \frac{3}{2}.$

Hence the equations of the asymptotes are

$$y = \tfrac{3}{4}x + \tfrac{1}{4}, \text{ i.e. } 4y = 3x + 1;$$

and $$y = -\tfrac{1}{2}x + \tfrac{3}{2}, \text{ i.e. } x + 2y = 3.$$

The equation of the hyperbola having asymptotes parallel to the above lines will be

$$(x + 2y + c_1)(3x - 4y + c_2) + c_3 = 0, \quad \dots\dots \quad \text{(iv)}$$

where c_1, c_2, c_3 are constants.

When $x = y$ in this equation,

$$(3x + c_1)(c_2 - x) + c_3 = 0,$$

i.e.
$$-3x^2 + x(3c_2 - c_1) + c_3 + c_1c_2 = 0.$$

But $x = 1$ satisfies this twice, therefore, using roots of a quadratic,

$$-3 + 3c_2 - c_1 + c_3 + c_1c_2 = 0, \quad \ldots\ldots \quad \text{(v)}$$

and
$$(3c_2 - c_1)/3 = 2 = \text{sum of roots},$$

i.e.
$$3c_2 - c_1 = 6. \quad \ldots\ldots\ldots \quad \text{(vi)}$$

Also, (3, 2) lies on the hyperbola,

$$\therefore\ (7 + c_1)(1 + c_2) + c_3 = 0,$$

i.e.
$$7 + c_1 + 7c_2 + c_3 + c_1c_2 = 0. \quad \ldots\ldots\ldots \quad \text{(vii)}$$

(vii) — (v) gives
$$10 + 2c_1 + 4c_2 = 0,$$

i.e.
$$c_1 + 2c_2 = -5. \quad \ldots\ldots\ldots \quad \text{(viii)}$$

(vi) + (viii) gives
$$5c_2 = 1, \quad \therefore\ c_2 = \tfrac{1}{5},$$
$$c_1 = -\tfrac{27}{5}.$$

From (v),
$$-3 + \tfrac{3}{5} + \tfrac{27}{5} + c_3 - \tfrac{27}{25} = 0,$$
$$\therefore\ c_3 = -\tfrac{48}{25}.$$

Hence the equation of the hyperbola is

$$(x + 2y - \tfrac{27}{5})(3x - 4y + \tfrac{1}{5}) - \tfrac{48}{25} = 0,$$

i.e.
$$(5x + 10y - 27)(15x - 20y + 1) - 48 = 0,$$

i.e.
$$75x^2 + 50xy - 200y^2 - 400x + 550y - 75 = 0,$$

i.e.
$$3x^2 + 2xy - 8y^2 - 16x + 22y - 3 = 0.$$

And the equations of the asymptotes are

$$5x + 10y = 27,$$
$$15x - 20y + 1 = 0.$$

Example 7 (L.U.).—Find the condition that the lines $y = mx$, $y = m'x$ may be conjugate diameters of the ellipse $ax^2 + 2hxy + by^2 = 1$.

Show that the equation of the equi-conjugate diameters of the ellipse

$$3x^2 + 4xy + 2y^2 = 1$$

is
$$5(3x^2 + 4xy + 2y^2) = 4(x^2 + y^2).$$

The equation of the ellipse is

$$ax^2 + 2hxy + by^2 = 1. \quad \ldots\ldots\ldots \quad \text{(i)}$$

Where the line
$$y = mx + c \quad \ldots\ldots\ldots \quad \text{(ii)}$$

cuts the ellipse (i), $ax^2 + 2hx(mx + c) + b(mx + c)^2 = 1$,

i.e.
$$x^2(a + 2hm + bm^2) + 2x(hc + bmc) + bc^2 - 1 = 0.$$

The mid-point (x, y) of the chord is given by

$$x = -\frac{(hc + bmc)}{bm^2 + 2hm + a},$$

and from (ii), $$y = \frac{-mhc - bm^2c + bm^2c + 2hmc + ac}{bm^2 + 2hm + a}$$

$$= \frac{hmc + ac}{bm^2 + 2hm + a}.$$

If this point lies on the line $y = m'x$,

$$\frac{hmc + ac}{bm^2 + 2hm + a} = -\frac{m'(bmc + hc)}{bm^2 + 2hm + a},$$

i.e. $$hm + a = -(bmm' + hm'),$$

i.e. $$bmm' + h(m + m') + a = 0,$$

i.e. the diameters are conjugate (bisect all chords parallel to each other) if

$$bmm' + h(m + m') + a = 0.$$

In the case of the ellipse

$$3x^2 + 4xy + 2y^2 = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

the condition that $y = mx$ and $y = m'x$ shall be conjugate diameters is, from this result,

$$2mm' + 2(m + m') + 3 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Consider the circle $$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

From (iii) — (v), the equation of the pair of lines through the origin passing through the points of intersection of the ellipse (iii) and the circle (v) is given by

$$y^2\left(2 - \frac{1}{r^2}\right) + 4xy + x^2\left(3 - \frac{1}{r^2}\right) = 0. \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

If the slopes of the lines (vi) be m and m', then

$$m + m' = -4\Big/\left(2 - \frac{1}{r^2}\right),$$

$$mm' = \left(3 - \frac{1}{r^2}\right)\Big/\left(2 - \frac{1}{r^2}\right).$$

Using these in (iv), the condition that the lines (vi) shall be equi-conjugate diameters (both are diameters of the circle (v) and therefore equal) is

$$\frac{2(3 - 1/r^2)}{(2 - 1/r^2)} - \frac{8}{(2 - 1/r^2)} + 3 = 0,$$

i.e. $$6 - \frac{2}{r^2} - 8 + 6 - \frac{3}{r^2} = 0,$$

i.e. $$4 = \frac{5}{r^2} \quad \therefore \frac{1}{r^2} = \frac{4}{5}.$$

Using this in equation (vi), the equation of the required lines is

$$y^2(2 - \tfrac{4}{5}) + 4xy + x^2(3 - \tfrac{4}{5}) = 0,$$

i.e.
$$3x^2 + 4xy + 2y^2 = \tfrac{4}{5}(x^2 + y^2),$$

i.e.
$$5(3x^2 + 4xy + 2y^2) = 4(x^2 + y^2).$$

EXAMPLES ON CHAPTER XI

All the following questions are taken from London University examination papers.

1. Prove that the equation of the chord joining the points on the conic

$$l = r(1 + e \cos \theta),$$

whose vectorial angles are α and β, is

$$l = r[e \cos \theta + \cos \{\theta - \tfrac{1}{2}(\alpha + \beta)\} \sec \tfrac{1}{2}(\alpha - \beta)].$$

PQ is a variable chord of an ellipse of which S is the focus, drawn parallel to the major axis. Show that the locus of the point in which the internal bisector of $\angle$PSQ meets PQ is a parabola with its vertex at S.

2. Find the polar equation of a conic referred to a focus as pole, and prove that, if S be the focus and PSP′ a focal chord, $1/\text{SP} + 1/\text{SP}' = 2/l$, where l is the length of the semi-latus rectum.

If the tangents at P and P′ meet in T, prove that $\tan \angle \text{PTP}' = 2a/\text{ST}$, where $2a$ is the length of the axis of the conic through S.

3. Show that the equation of the chord joining the points whose vectorial angles are $(\gamma + \delta)$, $(\gamma - \delta)$ on the conic $l = r(1 + e \cos \theta)$ is

$$l = r\{e \cos \theta + \sec \delta \cos (\theta - \gamma)\},$$

and deduce that the equation of the tangent at the point whose vectorial angle is α is

$$l = r\{e \cos \theta + \cos (\theta - \alpha)\}.$$

P and Q are two points with vectorial angles α, $\alpha - \frac{1}{3}\pi$, where $\frac{1}{2}\pi < \alpha < \pi$ on the conic $l = r(1 + e \cos \theta)$, with focus S. The tangent at P, and the chord PQ, meet the corresponding directrix at T and D respectively. SQ meets the tangent at P in R. Show that

$$\frac{1}{\text{SD}} + \frac{1}{\text{SR}} - \frac{\sqrt{3}}{\text{ST}} = \frac{1}{2l}.$$

4. Show that the equation of the tangent to the conic $l = r(1 + e \cos \theta)$, at the point $\theta = \alpha$, is given by

$$l = r\{\cos (\theta - \alpha) + e \cos \theta\}.$$

Show that the two conics

$$l\sqrt{3} = r(\sqrt{3} + \cos \theta) \quad \text{and} \quad l\sqrt{3} = 2r\{\sqrt{3} + \cos (\theta + \tfrac{1}{3}\pi)\}$$

touch where $\theta = \frac{1}{2}\pi$.

5. Find the equations of the asymptotes of the hyperbola

$$2x^2 - xy - 10y^2 + 4x - y + 6 = 0,$$

and draw a rough sketch showing the situation of the curve with regard to its asymptotes and the axes of co-ordinates.

By the use of tables, show that the eccentricity of the hyperbola is approximately 2·44.

6. An ellipse of eccentricity $\frac{1}{3}$ has one end of its major axis at the origin of rectangular co-ordinates and one end of the latus rectum through the focus nearer the origin, at the point (0, 2). If this focus be in the positive quadrant, find the co-ordinates of the other focus and the equation of the corresponding directrix.

7. Show that, for all values of α, the line whose equation is

$$x \cos \alpha + y \sin \alpha = \pm c\left(\frac{\lambda \sin 2\alpha + \cos 2\alpha}{\lambda^2 + 1}\right)^{\frac{1}{2}}$$

touches the conic $x^2 + 2\lambda xy - y^2 = c^2$.

Find the equations of the tangents to the hyperbolas $x^2 - y^2 = c^2$ and $2xy = c^2$, which are parallel to the line $x \cos \alpha + y \sin \alpha = 0$.

If the distances from the origin to these tangents are p and q, show that $p^4 + q^4$ is independent of α.

8. Show that the equation $x^2 - xy - 2y^2 - 5x + 7y = 0$ represents a hyperbola. Find the equation of its asymptotes and the co-ordinates of its centre.

Find the equation of the hyperbola which has the same asymptotes, and which has the origin, and the line $7y - 5x = 4$, as pole and polar.

9. Show that the locus of middle points of chords of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

which are parallel to the straight line $x/l = y/m$, is $lX + mY = 0$, where

$$X = ax + hy + g, \text{ and } Y = hx + by + f.$$

Show that $lX + mY = 0$ and $l_1X + m_1Y = 0$ are a pair of conjugate diameters of the conic if

$$all_1 + h(lm_1 + l_1m) + bmm_1 = 0.$$

10. Write down the equation of a hyperbola which has the straight lines $x + y + 1 = 0$ and $px + qy + r = 0$ for asymptotes.

Show that, if such a hyperbola can be drawn to touch the axes of co-ordinates, then $pq = r^2$.

11. Find separately the equations of the asymptotes of the conic

$$3x^2 + 19xy - 14y^2 - 8x + 13y - 5 = 0.$$

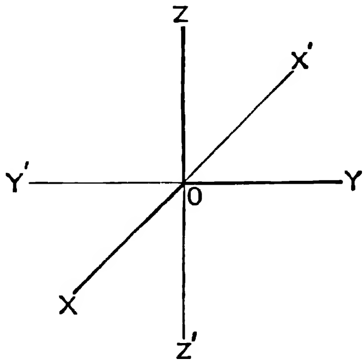
Prove that the equation of the straight line joining the points at a finite distance from the origin, in which the lines drawn through the point (1, 1) parallel to the asymptotes meet the conic, is

$$17x + 4y = 13.$$

CHAPTER XII

Co-ordinate Geometry in Three Dimensions—The Plane and the Straight Line

1. In three-dimensional geometry there are three mutually perpendicular axes OX, OY, OZ, with O the origin, where OZ is usually taken vertical and OX and OY horizontal, with OY, OZ in the plane of the paper, as shown in the diagram. XO, YO, ZO are produced to X′, Y′, Z′ respectively, thus forming eight compartments in space.



Any point in space has its position fixed if the lengths of the perpendiculars from it to the planes YOZ, ZOX, XOY be known, and these distances are the x, y, z co-ordinates respectively of the point in space.

Any point in the compartment XOYZ has all its co-ordinates positive. Any point in front of the plane ZOY has its x-co-ordinate positive, and any point behind the plane ZOY has its x-co-ordinate negative. Any point to the right of the plane ZOX has its y-co-ordinate positive, and any point to the left of this plane has its y-co-ordinate negative. Any point above the plane XOY has its z-co-ordinate positive, and any point below this plane has its z-co-ordinate negative.

Thus the point (1, 1, 1) will lie in the compartment XOYZ, the point (—1, 1, 1) in the compartment ZOYX′, the point (—1, —1, 1)

in the compartment ZOX'Y', the point $(-1, -1, -1)$ in the compartment Z'OX'Y', and so on.

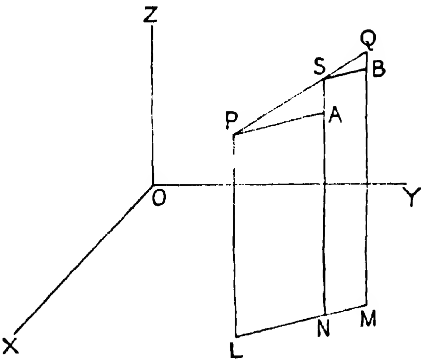
N.B.—When proving general theorems it is customary to use the compartment XOYZ, but the results will hold true for all compartments.

For all points in the plane YOZ the value of x is zero, and therefore the equation of the plane YOZ is $x = 0$.

Similarly, the equations of the planes ZOX and XOY are $y = 0$ and $z = 0$ respectively.

2. Theorem.—To find the co-ordinates of the point dividing the join of $P \equiv (x_1, y_1, z_1)$, $Q \equiv (x_2, y_2, z_2)$ in the ratio of $\lambda_2 : \lambda_1$.

Let the required point be $S \equiv (\bar{x}, \bar{y}, \bar{z})$. PL, SN, QM are the perpendiculars from P, S, Q respectively on the plane XOY. PA, SB are perpendiculars from P and S respectively on the lines SM, QN.



From the diagram,

$$SA = SN - AN = SN - PL = \bar{z} - z_1.$$
$$QB = QM - BM = QM - SN = z_2 - \bar{z}.$$

By similar triangles, $$\frac{SA}{QB} = \frac{PS}{SQ},$$

i.e. $$\frac{\bar{z} - z_1}{z_2 - \bar{z}} = \frac{\lambda_2}{\lambda_1},$$

i.e. $$\lambda_1\bar{z} - \lambda_1 z_1 = \lambda_2 z_2 - \lambda_2\bar{z}.$$

$$\therefore\ \bar{z} = \frac{\lambda_1 z_1 + \lambda_2 z_2}{\lambda_1 + \lambda_2}.$$

Similarly, it can be proved that

$$\bar{y} = \frac{\lambda_1 y_1 + \lambda_2 y_2}{\lambda_1 + \lambda_2}, \text{ and } \bar{z} = \frac{\lambda_1 z_1 + \lambda_2 z_2}{\lambda_1 + \lambda_2}.$$

Therefore the required point is

$$\left(\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2}, \frac{\lambda_1 y_1 + \lambda_2 y_2}{\lambda_1 + \lambda_2}, \frac{\lambda_1 z_1 + \lambda_2 z_2}{\lambda_1 + \lambda_2}\right).$$

As in plane co-ordinate geometry, the convention of sign with regard to internal and external division is positive for internal and negative for external division.

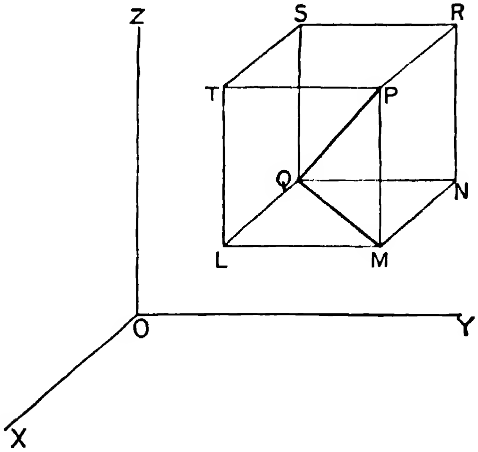
For the mid-point of the line PQ, $\lambda_1 = \lambda_2 = 1$, and the mid-point is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

3. Theorem.—To find the length of the line joining the points

$$\mathbf{P} \equiv (x_1, y_1, z_1), \quad \mathbf{Q} \equiv (x_2, y_2, z_2).$$

PRST, MNQL is a rectangular prism formed by drawing planes through P and Q parallel to the planes of reference XOY, ZOX, YOZ, as shown.



From the diagram it can be seen that

$$\mathrm{PM} = z_1 - z_2, \quad \mathrm{QL} = x_1 - x_2, \quad \mathrm{LM} = y_1 - y_2.$$

Using Pythagoras' theorem twice,

$$\begin{aligned} PQ^2 &= PM^2 + QM^2 \\ &= PM^2 + LM^2 + QL^2 \\ &= (z_1 - z_2)^2 + (y_1 - y_2)^2 + (x_1 - x_2)^2. \end{aligned}$$

$$\therefore\ PQ = \sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\}}.$$

In the case where Q is any point (x, y, z) on a sphere, radius r, centre P, then

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = PQ^2,$$

i.e.

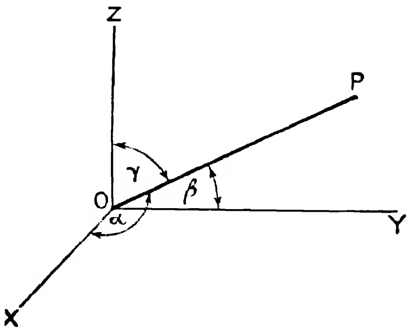
$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2,$$

which is the equation of the sphere, since Q is any point on it, and when $x_1 = y_1 = z_1 = 0$, P is at the origin, and the equation is

$$x^2 + y^2 + z^2 = r^2.$$

4. Direction cosines of a straight line.

Let $P \equiv (x, y, z)$ be any point in space such that $OP = r$, and let OP make angles α, β, γ with OX, OY, OZ respectively. The quantities $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are denoted by the letters l, m, n respectively, and



are known as the *direction cosines* of the straight line OP. It follows, since any two parallel lines must make the same angles with OX, OY, and OZ, that two parallel lines must have the same direction cosines.

Now, from the diagram,

$$\cos\alpha = x/r,\ \cos\beta = y/r,\ \cos\gamma = z/r,$$

i.e.

$$l = x/r,\ m = y/r,\ n = z/r.$$

$$\therefore\ l^2 + m^2 + n^2 = (x^2 + y^2 + z^2)/r^2.$$

But it has been proved that $r^2 = x^2 + y^2 + z^2$,

$$\therefore\ l^2 + m^2 + n^2 = 1.$$

This is an extremely important result and is the fundamental relationship between the *actual* direction cosines l, m, n of any straight line. (*N.B.*—In certain problems, quantities proportional to l, m, n can be used, and are known as *proportional* direction cosines, and it is thus advisable to refer to $\cos\alpha$, $\cos\beta$, $\cos\gamma$ as *actual* direction cosines.)

Also,

$$\begin{aligned}\sin^2\alpha + \sin^2\beta + \sin^2\gamma &= (1 - \cos^2\alpha) + (1 - \cos^2\beta) + (1 - \cos^2\gamma)\\ &= 3 - (l^2 + m^2 + n^2) = 3 - 1\\ &= 2.\end{aligned}$$

5. Theorem.—If the direction cosines of a straight line are proportional to f, g, h respectively, to find the actual direction cosines l, m, n.

By hypothesis, $l = kf$, $m = kg$, $n = kh$, where k is some constant.

$$\therefore\ l^2 + m^2 + n^2 = k^2(f^2 + g^2 + h^2),$$

i.e.
$$1 = k^2(f^2 + g^2 + h^2).$$

$$\therefore\ k = \frac{1}{\sqrt{(f^2 + g^2 + h^2)}} \quad \text{(choosing } k \text{ positive).}$$

$$\therefore\ l = \frac{f}{\sqrt{(f^2 + g^2 + h^2)}},\quad m = \frac{g}{\sqrt{(f^2 + g^2 + h^2)}},\quad n = \frac{h}{\sqrt{(f^2 + g^2 + h^2)}}.$$

6. Surfaces, &c.

It can be seen that any single relationship, $f(x, y, z) = 0$, between x, y, and z, will represent a surface, since a plane curve is always obtained by making z constant. Thus two simultaneous equations in x, y, and z will represent the intersection of two surfaces, i.e. a curve in space, and three simultaneous equations in x, y, and z will usually represent a number of points.

The Plane

7. Theorem.—To find the equation of the plane making intercepts a, b, c on the axes OX, OY, OZ respectively.

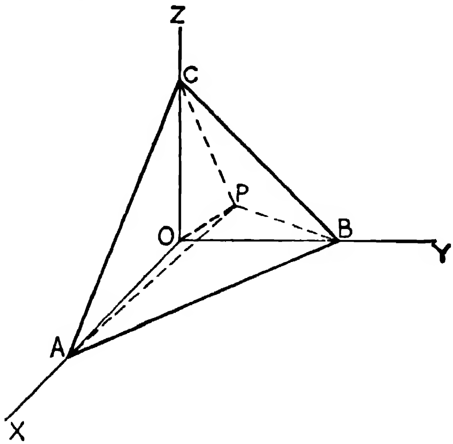
ABC is the given plane meeting OX, OY, OZ at A, B, C respectively,

$$\therefore\ \text{OA} = a,\ \text{OB} = b,\ \text{OC} = c.$$

$P \equiv (x, y, z)$ is any point on the plane ABC, and P is joined to O, A, B, C. Then

tetrahedron OABC = sum of tetrahedra OPBC, OPCA, OPBA,

i.e. $$\tfrac{1}{6}abc = \tfrac{1}{6}bcx + \tfrac{1}{6}cay + \tfrac{1}{6}abz.$$



Dividing through by $\frac{1}{6}abc$,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

which is the required equation, since P is any point on the plane.

It is to be noted that this equation is linear in x, y, and z, and therefore any linear equation in x, y, and z will represent a plane.

8. Theorem.—To find the *perpendicular form* of the equation of a plane, i.e. the equation of a plane, the perpendicular on which from the origin is of length p and has actual direction cosines l, m, n.

Let the given plane make intercepts a, b, c on OX, OY, OZ respectively. OD is the given perpendicular of length p on the plane ABC.

Using trigonometry, $p = al = bm = cn$,

$$\therefore\ a = p/l,\ b = p/m,\ c = p/n.$$

Now the equation of the plane ABC by the previous theorem is

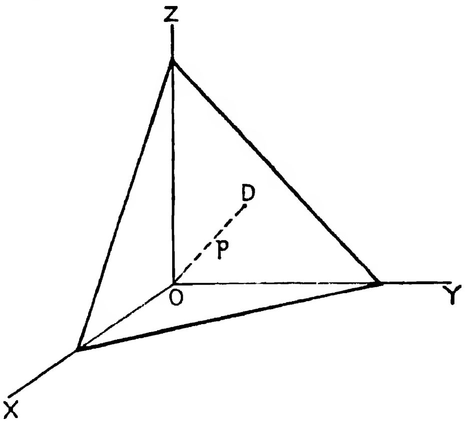
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Hence, using the values of a, b, c obtained, the equation of the plane is

$$\frac{lx}{p} + \frac{my}{p} + \frac{nz}{p} = 1,$$

i.e.
$$lx + my + nz = p.$$

N.B.—The perpendicular p from the origin on any plane is always taken to be *positive.*



From the previous results,

$$l = p/a, \quad m = p/b, \quad n = p/c,$$

$$\therefore \; l^2 + m^2 + n^2 = p^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

But $l^2 + m^2 + n^2 = 1$,

$$\therefore \; \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2},$$

a result which gives the length of the perpendicular from the origin on the plane making intercepts a, b, c on the axes OX, OY, OZ respectively.

Note.—The perpendicular form of the equation of a plane is the one most commonly required in the solution of problems.

9. Theorem.—To find the length of the perpendicular from the origin on the plane $ax + by + cz = d$.

The equation $\qquad ax + by + cz = d \qquad . \; . \; . \; . \; . \; . \; . \quad (1)$

is known as the *general* equation of a plane.

Let the perpendicular form of the equation of this plane be

$$lx + my + nz = p. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Since (1) and (2) represent the same plane, comparing coefficients,

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{p}{d} = k \qquad \text{(say).}$$

$$\therefore \; l = ak, \; m = bk, \; n = ck, \; p = dk.$$

$$\therefore \; l^2 + m^2 + n^2 = 1 = k^2(a^2 + b^2 + c^2).$$

$$\therefore \; k = \pm \frac{1}{\sqrt{(a^2 + b^2 + c^2)}}.$$

Hence
$$p = \pm \sqrt{(a^2 + b^2 + c^2)}$$

Since p must always be positive, it follows that the positive sign will be used when d is positive, and the negative sign when d is negative.

From this theorem it can be seen that the direction cosines of the perpendicular from the origin O to the plane $ax + by + cz = d$ are proportional to a, b, c, and the *actual* direction cosines of this perpendicular are l, m, n, given by

$$l = \pm \frac{a}{\sqrt{(a^2 + b^2 + c^2)}}, \quad m = \pm \frac{b}{\sqrt{(a^2 + b^2 + c^2)}}, \quad n = \pm \frac{c}{\sqrt{(a^2 + b^2 + c^2)}},$$

the positive sign being used when d is positive, and the negative sign when d is negative.

10. Theorem.—To find the equation of the plane perpendicular to the line whose direction cosines are l, m, n, that passes through the point (x_1, y_1, z_1).

The equation of the plane perpendicular to the given line and at a distance p from the origin O is

$$lx + my + nz = p. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

If this plane pass through (x_1, y_1, z_1),

$$lx_1 + my_1 + nz_1 = p. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Taking (3) — (4) to eliminate the unknown p, the required equation is

$$l(x - x_1) + m(y - y_1) + n(z - z_1) = 0.$$

N.B.—In this equation l, m, n need only be proportional direction cosines, as the equation can be divided through by any constant.

11. Theorem.—To find in determinant form the equation of the plane through the three given points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).

Let the required equation be

$$ax + by + cz + d = 0. \qquad \dots \dots \quad (5)$$

Since the three given points lie on the plane (5),

$$ax_1 + by_1 + cz_1 + d = 0, \qquad \dots \dots \quad (6)$$
$$ax_2 + by_2 + cz_2 + d = 0, \qquad \dots \dots \quad (7)$$
$$ax_3 + by_3 + cz_3 + d = 0. \qquad \dots \dots \quad (8)$$

Eliminating the unknown constants, a, b, c, d, between the equations (5), (6), (7), and (8), the equation of the plane is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

Example 1 (L.U.).—A plane passes through the three points whose rectangular co-ordinates are $(8, -2, 2)$, $(2, 1, -4)$ and $(2, 4, -6)$. Find the equation to this plane, the length of the perpendicular on it from the origin, and the direction cosines of that perpendicular.

Let the required equation be

$$ax + by + cz + d = 0. \qquad \dots \dots \dots \quad \text{(i)}$$

Since the given points lie on the plane,

$$8a - 2b + 2c + d = 0, \qquad \dots \dots \dots \quad \text{(ii)}$$
$$2a + b - 4c + d = 0, \qquad \dots \dots \dots \quad \text{(iii)}$$
$$2a + 4b - 6c + d = 0. \qquad \dots \dots \dots \quad \text{(iv)}$$

Eliminating a, b, c, d between these four equations, the equation of the required plane is

$$\begin{vmatrix} x & y & z & 1 \\ 8 & -2 & 2 & 1 \\ 2 & 1 & -4 & 1 \\ 2 & 4 & -6 & 1 \end{vmatrix} = 0,$$

i.e.

$$\begin{vmatrix} x-2 & y-4 & z+6 & 0 \\ 6 & -6 & 8 & 0 \\ 0 & -3 & 2 & 0 \\ 2 & 4 & -6 & 1 \end{vmatrix} = 0 \qquad \begin{matrix} (R_1 - R_4), \\ (R_2 - R_4), \\ \text{and } (R_3 - R_4), \end{matrix}$$

i.e.

$$\begin{vmatrix} x-2 & y-4 & z+6 \\ 6 & -6 & 8 \\ 0 & -3 & 2 \end{vmatrix} = 0,$$

i.e. expanding the determinant,

$$12(x-2)-12(y-4)-18(z+6)=0,$$

i.e. $$2(x-2)-2(y-4)-3(z+6)=0,$$

i.e. $$2x-2y-3z=14.$$

The length of the perpendicular from the origin on this line is

$$\frac{14}{\sqrt{\{2^2+(-2)^2+(-3)^2\}}}=\frac{14}{\sqrt{17}}=\frac{14\sqrt{17}}{17}=3\cdot395 \text{ to three decimal places.}$$

The direction cosines (actual) of this perpendicular which is normal to the plane are

$$\frac{2}{\sqrt{\{2^2+(-2)^2+(-3)^2\}}},\quad \frac{-2}{\sqrt{\{2^2+(-2)^2+(-3)^2\}}},\quad \frac{-3}{\sqrt{\{2^2+(-2)^2+(-3)^2\}}},$$

i.e. $$2/\sqrt{17},\quad -2/\sqrt{17},\quad -3/\sqrt{17},$$

i.e. $0\cdot4850$, $-0\cdot4850$, $-0\cdot7276$, to four decimal places.

THE STRAIGHT LINE

12. Since a straight line is the intersection of two planes, it follows that the equation of a straight line will be given by the simultaneous equations of two planes,

i.e. $$\left.\begin{aligned}a_1x+b_1y+c_1z+d_1=0\\ a_2x+b_2y+c_2z+d_2=0\end{aligned}\right\},$$

where the a's, b's, etc., are constants.

But this is not the best form of the equation of a straight line, and the equation most commonly used is the equation of a line passing through the point (x_1, y_1, z_1), and having direction cosines l, m, n, that now follows.

13. Theorem.—To find the equation of the line passing through the point (x_1, y_1, z_1) and having direction cosines l, m, n.

Let A be the point (x_1, y_1, z_1), and $P \equiv (x, y, z)$ be any point on the given line. AP is of length r, and the rectangular prism ABCD, PQRS is completed by means of planes through A and P, parallel to the planes of reference.

Since AB is parallel to OX,

$$AB/AP = l = \cos\alpha \quad (\text{where } \alpha = \angle PAB).$$

But $$AB = x - x_1,$$

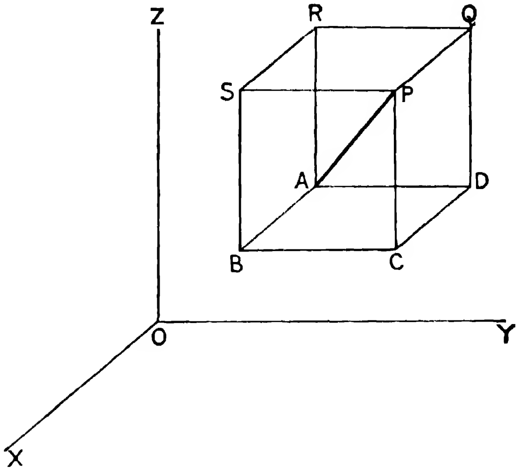
$$\therefore \frac{x-x_1}{r}=l, \text{ i.e. } \frac{x-x_1}{l}=r.$$

Similarly, $$\frac{y-y_1}{m}=r, \text{ and } \frac{z-z_1}{n}=r.$$

Hence the equations of the line, since P is any point on it, are given by

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r. \quad . \quad . \quad . \quad (9)$$

N.B.—The quantity r will be the actual length of AP if l, m, n are *actual* direction cosines.



The equation is the standard (or canonical) equation of a straight line, and if $x_1=y_1=z_1=0$, the equation is that of a line through the origin O, with direction cosines l, m, n, which is

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$

N.B.—In the case of the line (9), any point (x, y, z) on the line is given by

$$x=x_1+rl, \quad y=y_1+rm, \quad z=z_1+rn,$$

where r is the distance of the point in question from the point (x_1, y_1, z_1), if l, m, n are actual direction cosines.

14. Theorem.—To find the equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

Let the line have direction cosines l, m, n.

The equation of the line, since it passes through (x_1, y_1, z_1), is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}. \quad . \quad . \quad . \quad . \quad (10)$$

Since the point (x_2, y_2, z_2) lies on this line,

$$\frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n}. \quad . \quad . \quad . \quad . \quad (11)$$

From (10) $\div$ (11), to eliminate the unknowns l, m, n,

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1},$$

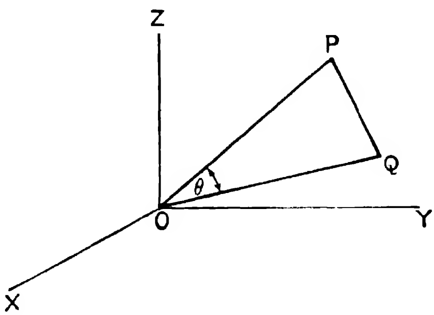
i.e. the required equation of the line is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}.$$

15. Theorem.—To find the angle θ between the two straight lines whose *actual* direction cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2).

OP, OQ are two straight lines with actual direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2) respectively, and $P \equiv (x_1, y_1, z_1)$, $Q \equiv (x_2, y_2, z_2)$.

Then $$\angle POQ = \theta.$$



Let $$OP = r_1, \text{ and } OQ = r_2.$$

Now $$PQ^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2.$$

Using the cosine rule,

$$PQ^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos \theta. \quad . \quad . \quad . \quad . \quad (12)$$

Also, it is known that

$$x_1 = r_1 l_1, \quad y_1 = r_1 m_1, \quad z_1 = r_1 n_1,$$
$$x_2 = r_2 l_2, \quad y_2 = r_2 m_2, \quad z_2 = r_2 n_2.$$
$$\therefore \ PQ^2 = (r_1 l_1 - r_2 l_2)^2 + (r_1 m_1 - r_2 m_2)^2 + (r_1 n_1 - r_2 n_2)^2.$$

Using this in (12),

$$(r_1 l_1 - r_2 l_2)^2 + (r_1 m_1 - r_2 m_2)^2 + (r_1 n_1 - r_2 n_2)^2$$
$$= r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta$$
$$= r_1^2(l_1^2 + m_1^2 + n_1^2) + r_2^2(l_2^2 + m_2^2 + n_2^2) - 2r_1 r_2 \cos\theta$$
$$(\text{since } l_1^2 + m_1^2 + n_1^2 = l_2^2 + m_2^2 + n_2^2 = 1).$$
$$\therefore \ -2r_1 r_2 l_1 l_2 - 2r_1 r_2 m_1 m_2 - 2r_1 r_2 n_1 n_2 = -2r_1 r_2 \cos\theta.$$
$$\therefore \ \cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2.$$

N.B.—It must be remembered that in this result (l_1, m_1, n_1), (l_2, m_2, n_2) are *actual* direction cosines.

When $\theta = 90^\circ$, $\cos\theta = 0$,

thus the condition for the lines to be perpendicular is

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0.$$

In this latter result the direction cosines involved need only be proportional.

16. Theorem.—To find the angle between the two planes

$$a_1 x + b_1 y + c_1 z = d_1, \quad a_2 x + b_2 y + c_2 z = d_2.$$

If θ be the angle between the two planes, then θ is the angle between the normals to the two planes.

The actual direction cosines of the normal to the first plane are

$$\frac{a_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}, \ \frac{b_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}, \ \frac{c_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}},$$

and the actual direction cosines of the normal to the second plane are

$$\frac{a_2}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}, \ \frac{b_2}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}, \ \frac{c_2}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}.$$

Using the previous theorem ($\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$) in this case,

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{\{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)\}}}.$$

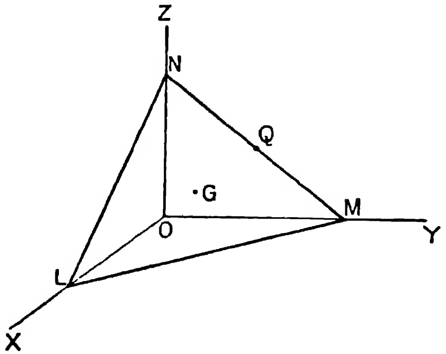
From this result it follows that, if the planes be at right angles,

$$a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

Example 2 (L.U.).—Prove that the equation of a plane is $lx + my + nz = p$, where l, m, n are the direction cosines of the perpendicular drawn from the origin on the plane, and p is the length of the perpendicular.

If G be the centroid of a triangle whose vertices are the intersections of this plane with the axes of co-ordinates, and the perpendicular through G to this plane meets the co-ordinate planes in A, B, C, prove that

$$\frac{1}{\text{GA}} + \frac{1}{\text{GB}} + \frac{1}{\text{GC}} = \frac{3}{p}.$$



The first part of the question has been proved as bookwork.

Using the equation $lx + my + nz = p$, it can be written

$$\frac{x}{p/l} + \frac{y}{p/m} + \frac{z}{p/n} = 1.$$

Thus the intercepts on OX, OY, OZ are p/l, p/m, p/n respectively.

Let the plane meet the axes in L, M, N respectively, and let Q be the mid-point of MN.

$$\text{L} \equiv (p/l, 0, 0), \quad \text{M} \equiv (0, p/m, 0), \quad \text{N} \equiv (0, 0, p/n),$$

$$\therefore \text{Q} \equiv (0, p/2m, p/2n).$$

Now $$\frac{\text{LG}}{\text{GQ}} = \frac{2}{1},$$

$$\therefore \text{G} \equiv (p/3l, p/3m, p/3n).$$

The equation of the line through G perpendicular to the plane LMN is

$$\frac{x - p/3l}{l} = \frac{y - p/3m}{m} = \frac{z - p/3n}{n} = r \qquad \text{(say)}.$$

When $x = 0$, $\qquad r = \text{GA}.$

But when $x = 0$,

$$r = -p/3l^2, \quad \therefore\ \text{GA} = -p/3l^2 = p/3l^2 \text{ numerically}.$$

Similarly, $\left.\begin{array}{l} \text{GB} = p/3m^2 \\ \text{GC} = p/3n^2 \end{array}\right\}$ numerically.

$$\therefore\ \frac{1}{\text{GA}} + \frac{1}{\text{GB}} + \frac{1}{\text{GC}} = \frac{3l^2}{p} + \frac{3m^2}{p} + \frac{3n^2}{p} = \frac{3(l^2 + m^2 + n^2)}{p}.$$

But $\qquad l^2 + m^2 + n^2 = 1,$

$$\therefore\ \frac{1}{\text{GA}} + \frac{1}{\text{GB}} + \frac{1}{\text{GC}} = \frac{3}{p}.$$

Example 3.—Find the equation of the plane through the point $(2, 3, -1)$ parallel to the plane $3x - 4y + 7z = 0$, and find the distance between the two planes. Find also the angle between this plane and the plane $8x + 3y - z = 2$.

The equation of the plane through (x_1, y_1, z_1), parallel to the plane

$$lx + my + nz = 0,$$

will be $l(x - x_1) + m(y - y_1) + n(z - z_1) = 0$, since both planes have the same normal. Therefore the equation of the required plane is

$$3(x - 2) - 4(y - 3) + 7(z + 1) = 0,$$

i.e. $\qquad 3x - 4y + 7z + 13 = 0.$

The distance between the two planes is the distance of the point $(2, 3, -1)$ from the plane $3x - 4y + 7z = 0$,

i.e. $$\pm\frac{3 \times 2 - 4(3) + 7(-1)}{\sqrt{\{3^2 + (-4)^2 + 7^2\}}} = \frac{13}{\sqrt{74}} \qquad \text{(using +ve sign)}.$$

If θ be the angle between the planes

$$a_1x + b_1y + c_1z = d_1 \quad \text{and} \quad a_2x + b_2y + c_2z = d_2,$$

then $$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{\{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)\}}}.$$

Hence, if θ be the angle between the planes

$$3x - 4y + 7z + 13 = 0 \text{ and } 8x + 3y - z = 2,$$

$$\cos\theta = \frac{8 \times 3 + 3 \times (-4) + (-1) \times 7}{\sqrt{[\{8^2 + 3^2 + (-1)^2\}\{3^2 + (-4)^2 + (7)^2\}]}} = \frac{5}{74} = 0{\cdot}06757,$$

$$\therefore\ \theta = 86^\circ\ 7'.$$

17. Theorem.—To prove that, if a plane cuts the join of P_1P_2 internally, and the co-ordinates of P_1 and P_2 be inserted in the equation of the plane, the resulting expressions have opposite signs.

Let P_1, P_2 be the points (x_1, y_1, z_1), (x_2, y_2, z_2) respectively, and the equation of the plane be

$$ax + by + cz + d = 0. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$P \equiv (\bar{x}, \bar{y}, \bar{z})$ is the point in which P_1P_2 cuts the plane (13).

Then

$$\bar{x} = \frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2}, \quad \bar{y} = \frac{\lambda_1 y_1 + \lambda_2 y_2}{\lambda_1 + \lambda_2}, \quad \bar{z} = \frac{\lambda_1 z_1 + \lambda_2 z_2}{\lambda_1 + \lambda_2},$$

where $\lambda_1 : \lambda_2 = P_2P : PP_1$.

Since P lies on the plane (13),

$$\frac{a(\lambda_1 x_1 + \lambda_2 x_2)}{\lambda_1 + \lambda_2} + \frac{b(\lambda_1 y_1 + \lambda_2 y_2)}{\lambda_1 + \lambda_2} + \frac{c(\lambda_1 z_1 + \lambda_2 z_2)}{\lambda_1 + \lambda_2} + d = 0,$$

i.e. $$\lambda_1(ax_1 + by_1 + cz_1 + d) + \lambda_2(ax_2 + by_2 + cz_2 + d) = 0,$$

$$\therefore \frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} = -\frac{\lambda_2}{\lambda_1}.$$

Since the division is internal, $\lambda_2 : \lambda_1$ is positive, and the theorem is proved.

Note.—This result is useful when considering the sign of the perpendicular from a point on to a plane.

18. Theorem.—To find the length of the perpendicular from the point (x_1, y_1, z_1) on the plane $lx + my + nz = p$.

Consider a plane through (x_1, y_1, z_1) parallel to the given plane

$$lx + my + nz = p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Since it has the same normal as the plane (14), its equation will be

$$lx + my + nz = p', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where p' is the perpendicular on this parallel plane from O, the origin. But the point (x_1, y_1, z_1) lies on the plane (15),

$$\therefore p' = lx_1 + my_1 + nz_1. \quad . \quad . \quad . \quad . \quad (16)$$

From a diagram it can be seen that the length of the perpendicular from the point (x_1, y_1, z_1) on the given plane is $p - p'$. Hence, using (16), the required length of perpendicular is

$$\boldsymbol{p - lx_1 - my_1 - nz_1}.$$

19. Theorem.—To find the length of the perpendicular from the point (x_1, y_1, z_1) on the plane

$$ax + by + cz + d = 0.$$

Let the perpendicular form of the equation of the plane

$$ax + by + cz + d = 0 \quad . \quad . \quad . \quad . \quad . \quad (17)$$

be

$$lx + my + nz = p. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Since equations (17) and (18) are equivalent,

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{p}{-d} = \pm\frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{(a^2 + b^2 + c^2)}} = \pm\frac{1}{\sqrt{(a^2 + b^2 + c^2)}}.$$

$$\therefore\ l = \pm\frac{a}{\sqrt{(a^2 + b^2 + c^2)}}, \quad m = \pm\frac{b}{\sqrt{(a^2 + b^2 + c^2)}},$$

$$n = \pm\frac{c}{\sqrt{(a^2 + b^2 + c^2)}}, \quad p = \mp\frac{d}{\sqrt{(a^2 + b^2 + c^2)}}. \qquad (19)$$

Now the length of perpendicular required, by the previous theorem, is $p - lx_1 - my_1 - nz_1$. Therefore, using the results in (19), the perpendicular length is

$$\mp\frac{d}{\sqrt{(a^2 + b^2 + c^2)}}$$
$$-\left\{\pm\frac{ax_1}{\sqrt{(a^2 + b^2 + c^2)}} \pm \frac{by_1}{\sqrt{(a^2 + b^2 + c^2)}} \pm \frac{cz_1}{\sqrt{(a^2 + b^2 + c^2)}}\right\}$$
$$= \pm\frac{(a_1x + by_1 + cz_1 + d)}{\sqrt{(a^2 + b^2 + c^2)}}.$$

Since the perpendicular from the origin on any plane has to be positive, the positive sign will be taken when d is positive, and the negative sign when d is negative.

20. Theorem.—To find the equation of the planes bisecting the angles between the planes

$$a_1x + b_1y + c_1z + d_1 = 0 \ \text{ and } \ a_2x + b_2y + c_2z + d_2 = 0.$$

From any point (x, y, z) on the required plane, the perpendiculars on the two given planes will be equal. Hence the required equation is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}} = \pm\frac{(a_2x + b_2y + c_2z + d_2)}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}.$$

The $\pm$ sign gives the two different planes bisecting the angle between the two given planes.

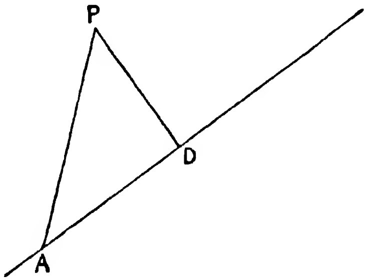
21. Theorem.—To find the length of the perpendicular from a given point to a given straight line.

$P \equiv (x_1, y_1, z_1)$ is the given point, and the equation of the straight line AD is

$$\frac{x - x_2}{l} = \frac{y - y_2}{m} = \frac{z - z_2}{n}, \quad \ldots \quad (20)$$

where $A \equiv (x_2, y_2, z_2)$ and PD is the required perpendicular.

Now $\qquad PA^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2.$



The equation of the plane through P perpendicular to the line (20) is

$$l(x - x_1) + m(y - y_1) + n(z - z_1) = 0, \quad \ldots \quad (21)$$

since the line (20) is normal to this plane.

Using the previous theorem, with l, m, n as *proportional* direction cosines,

$$AD = \pm \frac{l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)}{\sqrt{(l^2 + m^2 + n^2)}}.$$

By Pythagoras' theorem

$$PD^2 = PA^2 - AD^2.$$

$$\therefore \; PD^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - \left[\frac{l(x_1 - x_2) + m(y_1 - y_2) + n(z_1 - z_2)}{\sqrt{(l^2 + m^2 + n^2)}}\right]^2.$$

22. Theorem.—To find the condition that two given straight lines shall intersect.

Method (i).

Let the equations of the two straight lines be

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} = \rho_1 \quad \text{(say)}, \quad . \quad . \quad (22)$$

$$\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2} = \rho_2 \quad \text{(say)}. \quad . \quad . \quad (23)$$

From (22), $x = x_1 + \rho_1 l_1,\ y = y_1 + \rho_1 m_1,\ z = z_1 + \rho_1 n_1$. . (24)

From (23), $x = x_2 + \rho_2 l_2,\ y = y_2 + \rho_2 m_2,\ z = z_2 + \rho_2 n_2$. . (25)

If the lines (22) and (23) intersect, then, for some values of ρ_1 and ρ_2, the values of x, y, and z will be separately equal in equations (24) and (25). Thus

$$x_1 + \rho_1 l_1 = x_2 + \rho_2 l_2, \quad \text{i.e.} \quad x_1 - x_2 + \rho_1 l_1 - \rho_2 l_2 = 0. \quad (26)$$

$$y_1 + \rho_1 m_1 = y_2 + \rho_2 m_2, \quad \text{i.e.} \quad y_1 - y_2 + \rho_1 m_1 - \rho_2 m_2 = 0. \quad (27)$$

$$z_1 + \rho_1 n_1 = z_2 + \rho_2 n_2, \quad \text{i.e.} \quad z_1 - z_2 + \rho_1 n_1 - \rho_2 n_2 = 0. \quad (28)$$

Eliminating ρ_1 and $-\rho_2$ between the equations (26), (27), and (28), the required condition is

$$\begin{vmatrix} x_1 - x_2 & l_1 & l_2 \\ y_1 - y_2 & m_1 & m_2 \\ z_1 - z_2 & n_1 & n_2 \end{vmatrix} = 0.$$

Method (ii).

If the two lines (22) and (23) in Method (i) intersect, they will lie in a plane, and the points (x_1, y_1, z_1) and (x_2, y_2, z_2) will therefore lie in this plane.

Using l, m, n as the direction cosines of the normal to this plane, its equation will be

$$l(x - x_2) + m(y - y_2) + n(z - z_2) = 0, \quad . \quad . \quad (29)$$

since (x_2, y_2, z_2) lies on this plane.

But (x_1, y_1, z_1) also lies on this plane,

$$\therefore\ l(x_1 - x_2) + m(y_1 - y_2) + n(z_1 - z_2) = 0. \quad . \quad (30)$$

Since the lines (22) and (23) are perpendicular to the normal to the plane (29), since they lie in the plane,

$$ll_1 + mm_1 + nn_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$ll_2 + mm_2 + nn_2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Eliminating l, m, n between equations (30), (31), and (32), the required condition is

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Eliminating l, m, n between the equations (29), (31), and (32), the equation of the plane in which the lines (22) and (23) lie is

$$\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

23. Theorem.—To find the equation of the common perpendicular to two straight lines, and to find the length of this common perpendicular.

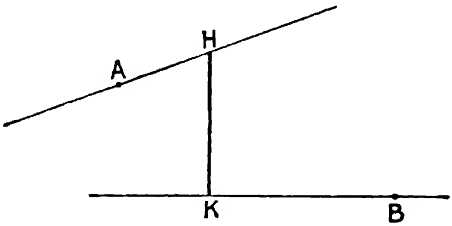
Let the equations of the lines be

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}, \quad . \quad . \quad . \quad . \quad (33)$$

$$\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}, \quad . \quad . \quad . \quad . \quad (34)$$

and HK the common perpendicular. A is the point (x_1, y_1, z_1) and B is the point (x_2, y_2, z_2).

Take L, M, N as the proportional direction cosines of HK.



Since HK is perpendicular to both the given lines,

$$Ll_1 + Mm_1 + Nn_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad (35)$$

$$Ll_2 + Mm_2 + Nn_2 = 0. \quad . \quad . \quad . \quad . \quad . \quad (36)$$

From (35) and (36),

$$\frac{L}{m_1 n_2 - m_2 n_1} = \frac{M}{l_2 n_1 - l_1 n_2} = \frac{N}{l_1 m_2 - l_2 m_1}. \quad . \quad . \quad (37)$$

Consider the plane AHK and take l, m, n as the direction cosines of its normal.

Since it passes through A, its equation is

$$l(x - x_1) + m(y - y_1) + n(z - z_1) = 0. \quad . \quad . \quad (38)$$

Since the line (33) lies in this plane, and is therefore perpendicular to the normal to it,

$$ll_1 + mm_1 + nn_1 = 0. \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Similarly, since HK lies in plane,

$$lL + mM + nN = 0. \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Using (37), and eliminating l, m, n between equations (38), (39), and (40), the equation of the plane AHK is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ m_1n_2 - m_2n_1 & l_2n_1 - l_1n_2 & l_1m_2 - l_2m_1 \end{vmatrix} = 0.$$

Similarly, the equation of the plane HKB is

$$\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ m_1n_2 - m_2n_1 & l_2n_1 - l_1n_2 & l_1m_2 - l_2m_1 \end{vmatrix} = 0,$$

and the line HK is the intersection of these two planes.

The equation of the plane through A perpendicular to HK is

$$L(x - x_1) + M(y - y_1) + N(z - z_1) = 0. \quad . \quad . \quad (41)$$

The length of HK will equal the length of the perpendicular from B on this plane (41). (KB is parallel to the plane (41).)

$$\therefore \text{ Length of HK} = \pm \frac{L(x_2 - x_1) + M(y_2 - y_1) + N(z_2 - z_1)}{\sqrt{(L^2 + M^2 + N^2)}}$$

$$= \pm \frac{(m_1n_2 - m_2n_1)(x_1 - x_2) - (l_1n_2 - l_2n_1)(y_1 - y_2) + (l_1m_2 - l_2m_1)(z_1 - z_2)}{\sqrt{\{(m_1n_2 - m_2n_1)^2 + (l_1n_2 - l_2n_1)^2 + (l_1m_2 - l_2m_1)^2\}}}$$

$$= \pm \frac{\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\{(m_1n_2 - m_2n_1)^2 + (l_1n_2 - l_2n_1)^2 + (l_1m_2 - l_2m_1)^2\}}}.$$

24. Theorem.—To find in the standard (canonical) form the equations of the line of intersection of two planes.

Let the equations of the planes be

$$a_1x + b_1y + c_1z + d_1 = 0, \quad . \quad . \quad . \quad . \quad (42)$$

$$a_2x + b_2y + c_2z + d_2 = 0. \quad . \quad . \quad . \quad . \quad (43)$$

Find by inspection a point (x_1, y_1, z_1) on the line of the intersection of the two planes. (It is usually best to choose the point in which the line cuts the plane XOY.)

Let the proportional direction cosines of the line be (l, m, n). Since the line lies in both planes (42) and (43),

$$la_1 + mb_1 + nc_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$la_2 + mb_2 + nc_2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Solving (44) and (45),

$$\frac{l}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - c_2a_1} = \frac{n}{a_1b_2 - a_2b_1}.$$

Thus the required equation is

$$\frac{x - x_1}{b_1c_2 - b_2c_1} = \frac{y - y_1}{c_1a_2 - c_2a_1} = \frac{z - z_1}{a_1b_2 - a_2b_1}.$$

Example 4 (L.U.).—Prove that the equation of the plane through the origin containing the line

$$\frac{x - a}{l} = \frac{y - b}{m} = \frac{z - c}{n}$$

is
$$x(bn - cm) + y(cl - an) + z(am - bl) = 0.$$

Deduce the equations of the line through the origin which intersects each of the lines

$$x = \tfrac{1}{2}(y - 1) = z + 1, \quad -(x + 2) = y = \tfrac{1}{2}(z - 2),$$

and find its direction cosines.

Part (*a*).—Let the direction cosines of the normal to the plane required be L, M, N (proportional).

Since the plane passes through the origin, its equation is

$$Lx + My + Nz = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (i)$$

But this plane contains the given line.

Therefore (a, b, c) lies on the plane, and the normal to this plane will be perpendicular to the given line.

Hence $$La + Mb + Nc = 0, \qquad \text{(ii)}$$
$$Ll + Mm + Nn = 0. \qquad \text{(iii)}$$

Solving (ii) and (iii), $$\frac{L}{bn - cm} = \frac{M}{cl - an} = \frac{N}{am - bl}.$$

Thus the equation of the required plane is

$$x(bn - cm) + y(cl - an) + z(am - bl) = 0.$$

Part (*b*).—Rewriting the equations of the lines in the form

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z+1}{1}, \qquad \text{(i)}$$

$$\frac{x+2}{-1} = \frac{y}{1} = \frac{z-2}{2}, \qquad \text{(ii)}$$

it can be seen from a diagram that the two planes through the lines (i) and (ii) and the origin will intersect in the required line.

Using Part (*a*), the equation of the plane through the line (i) and the origin is

$$x(1 + 2) + y(-1) + z(-1) = 0,$$

i.e. $$3x - y - z = 0. \qquad \text{(iii)}$$

Also the equation of the plane through the line (ii) and the origin is

$$x(-2) + y(-2 + 4) + z(-2) = 0,$$

i.e. $$x - y + z = 0. \qquad \text{(iv)}$$

Let (l, m, n) be the proportional direction cosines of the required line. Since it lies in planes (iii) and (iv), it is perpendicular to the normals to these planes,

$$\therefore\ 3l - m - n = 0,$$
$$l - m + n = 0,$$

from which $$\frac{l}{-2} = \frac{m}{-4} = \frac{n}{-2},$$

i.e. $$\frac{l}{1} = \frac{m}{2} = \frac{n}{1}.$$

Therefore the equation of the straight line is

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{1},$$

and its actual direction cosines are

$$\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}}.$$

25. Theorem.—To prove that, if $S_1 = 0$ and $S_2 = 0$ represent two planes, then, for all values of the constant λ, the equation $S_1 + \lambda S_2 = 0$

represents a plane passing through the line of intersection of the given planes. ($S_1 \equiv a_1x + b_1y + c_1z + d_1$, $S_2 \equiv a_2x + b_2y + c_2z + d_2$.)

The equation $S_1 + \lambda S_2 = 0$ will be linear in x and y, and will therefore represent a plane. Also, any point of intersection of the two planes $S_1 = 0$ and $S_2 = 0$ will satisfy the equation $S_1 + \lambda S_2 = 0$ for all values of λ, since it satisfies the equations $S_1 = 0$ and $S_2 = 0$.

Thus, **the equation $S_1 + \lambda S_2 = 0$ is the equation of a plane passing through the line of intersection of the planes $S_1 = 0$ and $S_2 = 0$.**

Note.—In any specific example where a definite plane is required, the value of λ can always be found by using the conditions stated.

26. Theorem.—To find the conditions that three given planes shall have a common line of intersection.

The equations of the three planes are taken as

$$a_1x + b_1y + c_1z + d_1 = 0, \quad \ldots \ldots \quad (46)$$
$$a_2x + b_2y + c_2z + d_2 = 0, \quad \ldots \ldots \quad (47)$$
$$a_3x + b_3y + c_3z + d_3 = 0. \quad \ldots \ldots \quad (48)$$

Any plane passing through the line of intersection of the planes (46) and (47) will be given by

$$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0. \quad (49)$$

For some value of the constant λ this plane will be equivalent to the plane (48), if the three planes (46), (47), (48) are to have a common line of intersection. Hence, comparing coefficients in equations (48) and (49),

$$\frac{a_1 + \lambda a_2}{a_3} = \frac{b_1 + \lambda b_2}{b_3} = \frac{c_1 + \lambda c_2}{c_3} = \frac{d_1 + \lambda d_2}{d_3} = -\mu \quad \text{(say)},$$

i.e.
$$a_1 + \lambda a_2 + \mu a_3 = 0, \quad \ldots \ldots \quad (50)$$
$$b_1 + \lambda b_2 + \mu b_3 = 0, \quad \ldots \ldots \quad (51)$$
$$c_1 + \lambda c_2 + \mu c_3 = 0, \quad \ldots \ldots \quad (52)$$
$$d_1 + \lambda d_2 + \mu d_3 = 0. \quad \ldots \ldots \quad (53)$$

Eliminating λ and μ between the equations (50), (51), (52), and (53), the required set of conditions is given by

$$\left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{matrix} \right\| = 0.$$

This is a special notation giving four conditions formed by omitting one row of the special type of determinant shown on the left-hand side. These four conditions, stated fully, are

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0. \quad \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0.$$

Example 5 (L.U.).—Find the equation of the plane through the origin and the line of intersection of the planes $3x - y + 2z = 4$ and $x + y + z = 1$.

Find also the equation of the plane through the origin perpendicular to the common line of these planes, and the perpendicular distance of the origin from this line.

The equations of the planes are

$$3x - y + 2z = 4, \quad \dots\dots\dots \quad \text{(i)}$$

$$x + y + z = 1. \quad \dots\dots\dots \quad \text{(ii)}$$

The equation of any plane passing through the line of intersection of the planes (i) and (ii) is given by

$$(3x - y + 2z - 4) + \lambda(x + y + z - 1) = 0.$$

If this plane pass through the origin,

$$-4 - \lambda = 0, \quad \therefore \lambda = -4.$$

Therefore the equation of the required plane is

$$3x - y + 2z - 4 - 4(x + y + z - 1) = 0.$$

i.e. $$-x - 5y - 2z = 0,$$

i.e. $$x + 5y + 2z = 0.$$

Let the equation of the plane through the origin perpendicular to the common line of planes (i) and (ii) be

$$lx + my + nz = 0. \quad \dots\dots\dots \quad \text{(iii)}$$

Then the direction cosines of the line given by equations (i) and (ii) will be l, m, n, since the line given by equations (i) and (ii) is normal to the plane (iii).

Since this line lies in the planes (i) and (ii),

$$3l - m + 2n = 0,$$

$$l + m + n = 0.$$

Solving these, $$\frac{l}{-3} = \frac{m}{-1} = \frac{n}{4}.$$

Therefore the required equation of the plane is

$$3x + y - 4z = 0. \quad \dots\dots\dots \quad \text{(iv)}$$

By inspection, a point on the line of intersection of planes (i) and (ii) is $(2, 0, -1)$.

It can be seen from a diagram that the perpendicular from the origin on the line of intersection of planes (i) and (ii) is the length of the line joining the origin to the point of intersection of the plane (iv), and the line of intersection of planes (i) and (ii).

Now the line of intersection of the planes (i) and (ii) is

$$\frac{x-2}{3} = \frac{y}{1} = \frac{z+1}{-4} = r \qquad \text{(say)}.$$

$$\left.\begin{aligned} \therefore\ x &= 3r+2, \\ y &= r, \\ z &= -4r-1. \end{aligned}\right\} \quad \dots\dots\dots \quad \text{(v)}$$

Substituting from (v) in (iv), for the point of intersection of the line and the plane,

$$9r + 6 + r + 16r + 4 = 0,$$

$$\therefore\ 26r = -10, \qquad \therefore\ r = -5/13.$$

Therefore the point of intersection is

$$\left(\frac{11}{13}, -\frac{5}{13}, \frac{7}{13}\right).$$

Therefore the required perpendicular distance is

$$\sqrt{\left(\frac{121}{169} + \frac{25}{169} + \frac{49}{169}\right)} = \sqrt{\left(\frac{195}{169}\right)} = \frac{\sqrt{195}}{13}.$$

Check.—Using the formula for perpendicular distance,

$$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2 - \left\{\frac{l(x_1-x_2) + m(y_1-y_2) + n(z_1-z^2)}{\sqrt{(l^2+m^2+n^2)}}\right\}^2}.$$

where $\quad x_2 = y_2 = z_2 = 0, \quad x_1 = 2, \quad y_1 = 0, \quad z_1 = -1,$

$$l = 3, \quad m = 1, \quad n = -4,$$

the perpendicular distance

$$= \sqrt{5 - \left(\frac{6+4}{\sqrt{26}}\right)^2}$$

$$= \sqrt{\left(5 - \frac{100}{26}\right)} = \sqrt{\left(\frac{15}{13}\right)} = \frac{\sqrt{195}}{13}.$$

Example 6 (L.U.).—If l be the line $\dfrac{x-1}{2} = \dfrac{y}{-1} = \dfrac{z+2}{1}$, find

(i) the direction cosines of its projection on the plane $2x + y - 3z = 4$;

(ii) the equation of the plane through l, which is parallel to the line of intersection of the planes

$$2x + 5y + 3z = 4, \text{ and } x - y - 5z = 6.$$

(i) Let l_1, m_1, n_1 be the proportional direction cosines of the normal to the plane containing the line l and perpendicular to the plane

$$2x + y - 3z = 4. \quad \dots\dots\dots \quad \text{(i)}$$

Since the normal (l_1, m_1, n_1) is perpendicular to the line

$$\frac{x-1}{2} = \frac{y}{-1} = \frac{z+2}{1}, \quad \ldots \ldots \quad \text{(ii)}$$

it follows that $$2l_1 - m_1 + n_1 = 0. \quad \ldots \ldots \quad \text{(iii)}$$

Also, since the required line is perpendicular to the plane (i), it follows that

$$2l_1 + m_1 - 3n_1 = 0. \quad \ldots \ldots \quad \text{(iv)}$$

Solving (iii) and (iv), $$\frac{l_1}{2} = -\frac{m_1}{-8} = \frac{n_1}{4},$$

i.e. $$\frac{l_1}{1} = \frac{m_1}{4} = \frac{n_1}{2}.$$

If l_2, m_2, n_2 be the direction cosines of the projection of l on the plane (i), since it lies in the plane (i),

$$2l_2 + m_2 - 3n_2 = 0. \quad \ldots \ldots \quad \text{(v)}$$

Also, since the projection is perpendicular to the line having direction cosines $(l_1, m_1, n_1) = (1, 4, 2)$,

$$l_2 + 4m_2 + 2n_2 = 0. \quad \ldots \ldots \quad \text{(vi)}$$

Solving (v) and (vi), $$\frac{l_2}{14} = -\frac{m_2}{7} = \frac{n_2}{7},$$

$$\therefore \frac{l_2}{2} = \frac{m_2}{-1} = \frac{n_2}{1}.$$

Therefore the actual direction cosines of the projection of l are

$$\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}.$$

(ii) The line l can be written

$$\left.\begin{aligned} x-1 &= -2y \\ x-1 &= 2(z+2) \end{aligned}\right\}, \quad \text{i.e.} \quad \left.\begin{aligned} x+2y-1 &= 0 \\ x-2z-5 &= 0 \end{aligned}\right\}. \quad \ldots \quad \text{(i)}$$

Any plane through the line (i) will be given by

$$x + 2y - 1 + \lambda(x - 2z - 5) = 0. \quad \ldots \ldots \quad \text{(ii)}$$

If L, M, N be the proportional direction cosines of the line of intersection of the planes,

$$2x + 5y + 3z = 4, \quad \ldots \ldots \quad \text{(iii)}$$
$$x - y - 5z = 6, \quad \ldots \ldots \quad \text{(iv)}$$

since it lies in the planes (iii) and (iv),

$$2L + 5M + 3N = 0, \quad \ldots \ldots \quad \text{(v)}$$
$$L - M - 5N = 0. \quad \ldots \ldots \quad \text{(vi)}$$

Solving (v) and (vi), $$\frac{L}{-22} = \frac{-M}{-13} = \frac{N}{-7},$$

i.e. $$\frac{L}{22} = \frac{M}{-13} = \frac{N}{7}.$$

If the plane (ii) is to be parallel to the intersection of the planes (iii) and (iv), the normal to the plane (ii) is perpendicular to this intersection.

$$\therefore\ (1 + \lambda)L + 2M - 2\lambda N = 0,$$

i.e. $$22(1 + \lambda) - 26 - 14\lambda = 0,$$

$$\therefore\ 8\lambda = 4, \quad \therefore\ \lambda = \tfrac{1}{2}.$$

Hence the required plane is

$$x + 2y - 1 + \tfrac{1}{2}(x - 2z - 5) = 0,$$

i.e. $$2x + 4y - 2 + x - 2z - 5 = 0,$$

i.e. $$3x + 4y - 2z = 7.$$

27. Projection of an area on a plane.

If a given plane area be S square units and its plane makes an angle θ with a given plane, then its projected area on the given plane is known to be $S \cos \theta$.

Using the notation S_{yz}, S_{zx}, S_{xy} for the projections of the plane S on the planes YOZ, ZOX, XOY respectively, if the normal to the plane S have actual direction cosines l, m, n, then, since $l = \cos \alpha$, where α is the angle the normal to S makes with OX, it follows that

$$S_{yz} = lS.$$

Similarly, $$S_{zx} = mS,$$

and $$S_{xy} = nS.$$

Squaring and adding these results,

$$(S_{yz})^2 + (S_{zx})^2 + (S_{xy})^2 = S^2(l^2 + m^2 + n^2)$$
$$= S^2.$$

$$\therefore\ \boldsymbol{S^2 = S_{yz}{}^2 + S_{zx}{}^2 + S_{xy}{}^2.}$$

Also, if the plane S make an angle ϕ with a second plane, the normal to which has direction cosines l_1, m_1, n_1, then

$$\cos \phi = ll_1 + mm_1 + nn_1.$$

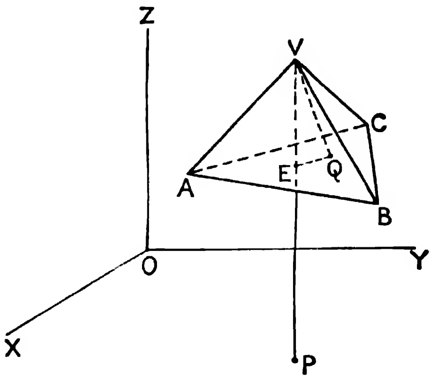
$$\therefore\ S \cos \phi = (Sl)l_1 + (Sm)m_1 + (Sn)n_1$$
$$= l_1 S_{yz} + m_1 S_{zx} + n_1 S_{xy},$$

i.e. the projection of the plane S on the second plane is equal to

$$\boldsymbol{l_1 S_{yz} + m_1 S_{zx} + n_1 S_{xy}.}$$

28. Theorem.—To find the volume of the tetrahedron VABC, where $V \equiv (x_4, y_4, z_4)$, $A \equiv (x_1, y_1, z_1)$, $B \equiv (x_2, y_2, z_2)$, and $C \equiv (x_3, y_3, z_3)$.

Method (i).—VP is the line perpendicular to XOY meeting ABC at E. VQ is the perpendicular on ABC of length h and VE $= h_1$.



The equation of the plane ABC is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

Now $E \equiv (x_4, y_4, z_4 - h_1)$ lies on the plane ABC,

$$\therefore \begin{vmatrix} x_4 & y_4 & z_4 - h_1 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0,$$

i.e.
$$\begin{vmatrix} x_4 & y_4 & z_4 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} - h_1 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

$$\therefore h_1 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \text{ (neglecting sign).} \quad . \quad . \quad (54)$$

Now the volume of the tetrahedron VABC

$$= \tfrac{1}{3}\triangle ABC \times VQ = \tfrac{1}{3}\triangle ABC \times h_1 \cos\theta$$

$$= \tfrac{1}{3}h_1 \times \triangle ABC \cos\theta$$

$$= \tfrac{1}{3}h_1 \times \text{the projection of } \triangle ABC \text{ on the plane XOY}$$

(since $\theta = \angle EVQ =$ angle between planes ABC, XOY)

$$= \tfrac{1}{3}h_1 \times \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \tfrac{1}{6}h_1 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Therefore, using the result (54), the volume of the tetrahedron VABC is

$$\tfrac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

Method (ii).—The equation of the plane ABC is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0. \quad \ldots\ldots \quad (55)$$

The perpendicular VQ from V on this plane is given by

$$\begin{vmatrix} x_4 & y_4 & z_4 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}$$

divided by the root of the sum of the squares of the minors of x, y, z of the determinant in (55).

These minors are

$$\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}; \quad -\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}; \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix};$$

i.e. with the previous notation,

$$2S_{yz},\ 2S_{zx},\ 2S_{xy},$$

where S is the area of $\triangle ABC$.

Using $S^2 = S_{yz}^2 + S_{zx}^2 + S_{xy}^2$, the denominator of the fraction for the perpendicular length from V will be $2S$.

$$\therefore\ \mathrm{VQ} \times 2S = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

(interchanging rows in the previous determinant and neglecting the sign).

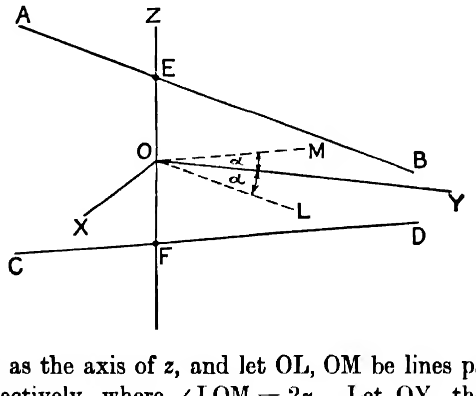
Therefore the volume of the tetrahedron $= \frac{1}{3}\mathrm{VQ}\,.\,S$

$$= \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

29. Theorem.—To show that, by a suitable choice of axes, the equations of any two non-intersecting straight lines may be expressed in the form

$$y = mx, \quad z = c;$$
$$y = -mx, \quad z = -c.$$

Let AB, CD be the two given lines, EF their common perpendicular of length $2c$, and choose the origin as O, the mid-point of EF. Choose



OE produced as the axis of z, and let OL, OM be lines parallel to AB and CD respectively, where $\angle$LOM $= 2\alpha$. Let OY, the bisector of $\angle$MOL, be the y-axis, and OX, perpendicular to the plane YOZ, be the x-axis.

For any point on OL $\qquad y = x \cot \alpha,$

i.e. the line OL is the line given by

$$y = mx,\ z = 0, \text{ where } m = \cot \alpha.$$

Similarly, the line OM is given by

$$y = -mx, \ z = 0.$$

But AB is parallel to OL and distant $+c$ from it. Hence the equations of AB, with respect to the stated axes and origin, are

$$y = mx, \ z = c;$$

and, similarly, the equations for CD are

$$y = -mx, \ z = -c.$$

Example 7 (L.U.).—Show that, by a suitable choice of axes, the equations of any two non-intersecting straight lines may be expressed in the form

$$y = mx, \quad z = c; \quad y = -mx, \quad z = -c.$$

Lines are drawn to intersect these two lines and to make a constant angle with the z-axis. Show that, if $m < 1$, the locus of their mid-points is an ellipse of eccentricity $\sqrt{(1 - m^4)}$.

The first part of the question was proved in the previous theorem.

Part (ii).—Let a point on the first line have an x-co-ordinate of x_1, then the point will be (x_1, mx_1, c). If x_2 be the x-co-ordinate of a point on the second line, the point will be $(x_2, -mx_2, -c)$.

The mid-point $(\bar{x}, \bar{y}, \bar{z})$ of the join of these two points will be given by

$$\bar{x} = \tfrac{1}{2}(x_1 + x_2), \ \bar{y} = \tfrac{1}{2}m(x_1 - x_2), \ \bar{z} = 0. \quad . \quad . \quad . \quad . \quad \text{(i)}$$

The direction cosines of the joins of these two points will be proportional to

$$(x_1 - x_2), \ m(x_1 + x_2), \ 2c,$$

i.e. using (i), $\quad 2\bar{y}/m, \ 2\bar{x}m, \ 2c,$

i.e. $\quad \bar{y}/m, \ \bar{x}m, \ c.$

Thus the actual direction cosines of the join will be

$$\frac{\bar{y}/m}{\sqrt{(\bar{y}^2/m^2 + \bar{x}^2m^2 + c^2)}}, \quad \frac{\bar{x}m}{\sqrt{(\bar{y}^2/m^2 + \bar{x}^2m^2 + c^2)}}, \quad \frac{c}{\sqrt{(\bar{y}^2/m^2 + \bar{x}^2m^2 + c^2)}}.$$

The direction cosines of the z-axis are (0, 0, 1), therefore if α be the constant angle between the z-axis and the join of the two points,

$$\cos\alpha = 0 + 0 + \frac{}{\sqrt{(\bar{y}^2/m^2 + \bar{x}^2m^2 + c^2)}} = \frac{}{\sqrt{(\bar{y}^2/m^2 + \bar{x}^2m^2 + c^2)}},$$

i.e. $\quad \cos^2\alpha(\bar{y}^2/m^2 + \bar{x}^2m^2 + c^2) = c^2,$

i.e. $\quad \bar{x}^2m^2 + \bar{y}^2/m^2 = c^2(1 - \cos^2\alpha)/\cos^2\alpha$

$$= k^2 \qquad \text{(where } k = c\tan\alpha = \text{constant)},$$

i.e. $$\frac{\bar{x}^2}{k^2/m^2} + \frac{\bar{y}^2}{m^2k^2} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

The equation (i) shows that the required locus lies in the plane XOY, and converting to running co-ordinates, the equation of the locus in the plane XOY will be

$$\frac{x^2}{k^2/m^2} + \frac{y^2}{k^2m^2} = 1.$$

This shows that the locus is an *ellipse in the plane* XOY, and if e be the eccentricity of the ellipse,

$$m^2k^2 = \frac{k^2}{m^2}(1 - e^2),$$

i.e. $$m^4 = 1 - e^2, \quad \text{i.e. } e^2 = 1 - m^4.$$

$$\therefore\ e = \sqrt{(1 - m^4)}.$$

EXAMPLES ON CHAPTER XII

All the following examples are taken from London University examination papers.

1. Planes are drawn so as to make an angle $\pi/3$ with the line $x = y = z$, and an angle $\pi/4$ with the line $x = 0$, $y - z = 0$. Show that all these planes are inclined at an angle $\pi/3$ with the plane $x = 0$. Find the equations of the two planes of the above family which are distant three units from the point (2, 1, 1).

2. If the direction cosines of the vertical are proportional to $(+1, -2, -1)$, and the vertical through the origin cuts the plane $x - 3y + 2z = 10$ at the point P, find the equations of the line of greatest slope through P in the plane.

3. Find the angle between the two straight lines whose direction cosines are (l, m, n) and (l', m', n').

Find the equations of the two straight lines through the origin, each of which intersects the straight line $\dfrac{x-3}{2} = \dfrac{y-3}{1} = \dfrac{z}{1}$, and is inclined at an angle of 60° to it.

4. Find the equations of the straight line which is the projection on the plane $3x + 2y + z = 0$, of the straight line which is the intersection of the planes $3x - y + 2z = 1$ and $x + 2y - z = 2$.

Show that the point $(-1, 1, 1)$ lies on this projection, and express the equation of this projection in the form

$$\frac{x+1}{l} = \frac{y-1}{m} = \frac{z-1}{n}.$$

5. A fixed line AEB passes through the points $A \equiv (2, 2, 2)$, $B \equiv (-1, -1, -3)$, and a variable line CFD passes through the point $C \equiv (2, 3, 1)$, and moves so that the direction cosines of the common perpendicular EF are always proportional to $(0, 5, -3)$. Find the equation of the locus of F, and the length of EF.

6. The direction cosines of two intersecting lines are (l_1, m_1, n_1) and (l_2, m_2, n_2). Show that the direction cosines of any line lying in their plane and passing through their point of intersection are proportional to $l_1 + \lambda l_2$, $m_1 + \lambda m_2$, $n_1 + \lambda n_2$, where λ is an appropriate constant.

QP, RP are two lines through a point P whose direction cosines are proportional to $(1, 1, -2)$ and $(1, -1, 1)$ respectively. Find the equation of the plane through the origin which is perpendicular to the plane RPQ and parallel to the line QP. If the co-ordinates of P are $(-1, 1, 1)$, find the co-ordinates of the foot of the perpendicular from P on to this plane.

7. Find the equations, in the standard form, of the projection of the straight line $\dfrac{x+1}{3} = \dfrac{y-2}{2} = \dfrac{z-3}{-1}$ on the plane $x + y + 2z - 4 = 0$.

Find the projection of the point $(-1, 2, 3)$ on this plane.

8. Show that the angle θ between the lines

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \text{ and } \frac{x-\alpha}{\lambda} = \frac{y-\beta}{\mu} = \frac{z-\gamma}{\nu}$$

is given by the equation

$$\cos\theta = \frac{l\lambda + m\mu + n\nu}{\sqrt{\{(l^2+m^2+n^2)(\lambda^2+\mu^2+\nu^2)\}}}.$$

Find the equations of the lines through the origin which meet both the z-axis and the line $3x = 4z$, $y = 0$ at $60°$.

9. Find the angle between two lines whose direction cosines are (l, m, n) and (l_1, m_1, n_1).

Find the equations of the lines that intersect the given line

$$\frac{x+5}{2} = \frac{y+6}{2} = \frac{z+7}{1}$$

at $60°$, and which lie in the plane $x + y + 2z + 1 = 0$.

10. Find a formula for the shortest distance between the lines whose equations are

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \text{ and } \frac{x-a_1}{l_1} = \frac{y-b_1}{m_1} = \frac{z-c_1}{n_1}.$$

Find the magnitude and direction of the shortest distance between the lines

$$\frac{x+7}{-8} = \frac{y-5}{3} = \frac{z-4}{1} \text{ and } \frac{x+4}{4} = \frac{y}{3} = \frac{z-19}{-2}.$$

11. Show that the equations of two skew lines can be written in the form

$$y = mx,\ z = c;\quad y = -mx,\ z = -c.$$

The shortest distance between two skew lines is AB. P is a variable point on the line on which A lies, and Q is a variable point on the line on which B lies, so that AQ and BP are at right angles. Prove that the locus of the mid-point of PQ is a hyperbola, whose asymptotes are parallel to the given skew lines, provided that the latter are not perpendicular.

12. Prove that the angle θ between two lines whose direction cosines are (l, m, n) and (l_1, m_1, n_1) is given by $\cos\theta = ll_1 + mm_1 + nn_1$.

The vertices A, B, C, D of a tetrahedron have the co-ordinates $(0, 0, 0)$,

(1, 2, 3), (4, −1, 1), (−3, 1, −2) respectively. Find, correct to $30''$, (i) the angle between the lines AB and CD; (ii) the angle between the planes ABC and ABD.

13. Show how to find the equation of a plane through three points whose co-ordinates are given.

Find the equations of the two planes through the two points (0, 4, −3), (6, −4, 3), other than the plane through the origin, which cut off from the axes intercepts whose sum is zero.

14. Prove that the angle θ between two lines whose direction cosines are (l, m, n) and (l_1, m_1, n_1) is given by

$$\cos\theta = ll_1 + mm_1 + nn_1.$$

Find the angle between the common line of the planes $x + y - z = 1$ and $2x - 3y + z = 2$, and the line joining the points (3, −1, 2) and (4, 0, −1). Find also the equation of a line through the origin which is perpendicular to both the above lines.

15. Find the equations of the shortest distance between the lines

$$\left.\begin{array}{l} y = 3x + 5 \\ z = 2x - 3 \end{array}\right\} \text{ and } \left.\begin{array}{l} y = 4x - 7 \\ z = 5x + 10 \end{array}\right\}.$$

Calculate also the length of the shortest distance, and the co-ordinates of the points in which it meets each of the lines.

16. A plane passes through the points (1, 2, 3), (−1, 2, 0), and (2, −1, −1). Find its equation, the area of the triangle formed by the lines in which it intersects the co-ordinate planes, and the co-ordinates of the foot of the perpendicular from the origin to the plane.

17. Find the equations to the line through the origin, perpendicular to and intersecting the line $x + 2y + 3z + 4 = 0$, $2x + 3y + 4z + 5 = 0$, and determine the co-ordinates of the point in which these lines intersect.

18. Show that the lines

$$(a)\ \frac{x-2}{2} = \frac{y-3}{-1} = \frac{z+4}{3} \quad \text{and} \quad (b)\ \frac{x-3}{1} = \frac{y+1}{3} = \frac{z-1}{-2}$$

intersect.

Find their common point and the equation of the plane containing them.

Find also the equation of the plane perpendicular to this plane and passing through the line (a).

19. Show that the shortest distance between the lines

$$\frac{x}{2} = \frac{y}{-3} = \frac{z}{1} \text{ and } \frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2} \text{ is } \tfrac{1}{3}\sqrt{3}.$$

Show also that the shortest distance lies along the line of intersection of the planes

$$4x + y - 5z = 0,$$
$$7x + y - 8z = 31.$$

20. Find the length of the shortest distance between the lines whose equations are

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n} \text{ and } \frac{x-a_1}{l_1}=\frac{y-b_1}{m_1}=\frac{z-c_1}{n_1}.$$

Find the length and the equations of the shortest distance between the lines whose equations are

$$x+y=0,\ z=4 \text{ and } \frac{x-1}{4}=\frac{y-2}{3}=\frac{z-36}{-6}.$$

21. Show that, by a suitable choice of axes, the equations of any two lines L and L_1 can be put into the form

$$L,\ y=x\tan\alpha,\ z=c, \quad \text{and} \quad L_1,\ y=-x\tan\alpha,\ z=-c.$$

Two lines L and L_1, whose equations are in the above form, are met by their shortest distance at the points A on L and B on L_1, and points P and Q taken on L and L_1 respectively, so that PQ = AP + BQ. Show that PQ makes an angle of $\pi/2-\alpha$ with the y-axis and that the locus of the point of intersection of PQ and the plane $y=0$ is a circle on AB as diameter.

22. Find the point of intersection of the plane π, whose equation is $2x-y-z+3=0$, with the line L, whose equations are $2x+y-4=0$, $y+2z-8=0$.

Find the equations defining L_1, the projection of the line L on the plane π, and also the angle between L and L_1. If L_1 is the line of greatest slope in the plane π, and the angle between L_1 and the vertical is 60°, find the direction cosines of either possible vertical.

23. Find the equations of the line through the point (1, 2, 3) which intersects at right angles the line

$$\frac{x-2}{1}=\frac{y-1}{2}=\frac{z}{3}.$$

Determine the co-ordinates of their point of intersection and the equation of the plane containing them.

24. The equations of two lines are

$$x=y+2a=6(z-a) \text{ and } x+a=2y=-12z.$$

Find the equations of the two planes, each containing one of the two lines and parallel to the other, and hence determine the shortest distance between the two lines.

25. Show that the planes

$$x+2y-z=0,\quad 3x-4y+z=3,\quad \text{and}\quad 4x+3y-2z=24$$

form a triangular prism, and that the lengths of the sides of a normal cross-section are in the ratio $\sqrt{13}:\sqrt{58}:5\sqrt{3}$.

26. The points A $(2a, 0, 0)$, B $(0, 2b, 0)$, C $(0, 0, 0)$, and D (a, b, c) are the corners of a tetrahedron. Find the equation to the line that intersects and is perpendicular to AC and BD. Hence, or otherwise, determine the shortest distance between the two lines. Also find the volume of the tetrahedron.

27. Find the angle between the lines

$$\left.\begin{aligned} x + y + z &= 6 \\ 3x - y + 4z &= 13 \end{aligned}\right\} \text{ and } \left.\begin{aligned} 2x - y + 2z &= 0 \\ x - 2z &= 0 \end{aligned}\right\}.$$

Find also the equation of the plane through the first line parallel to the second.

28. Show that the line whose equations are

$$6x + 4y - 5z = 4, \; x - 5y + 2z = 12$$

intersects the line $$\frac{x-9}{2} = \frac{y+4}{-1} = \frac{z-5}{1}.$$

Find the equation of the plane containing these lines, and also the angle between the lines.

29. Prove that the equation of the plane which passes through the point (α, β, γ) and the line $x = ay + b = cz + d$ is

$$\begin{vmatrix} x & ay + b & cz + d \\ \alpha & a\beta + b & c\gamma + d \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Hence find, in its simplest form, the equation of the plane which passes through the point (1, 1, 1) and the line

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}.$$

Find also the perpendicular distance of the point (1, 1, 1) from the given line.

CHAPTER XIII

The General Conicoid—Sphere, Cone, etc.

The General Equation of the Second Degree

1. The general equation of the second degree in x, y, and z is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0,$$

and represents a surface. The equation contains ten apparent constants, but by dividing through by one of these, it can be seen that there are only nine *independent* constants. Thus, the surface represented by the equation can be made to satisfy nine independent conditions, such as passing through nine points, etc.

2. Theorem.—To find the points of intersection of the straight line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n},$$

and the surface represented by the general equation of the second degree in x, y, and z.

The surface is given by the equation

$$\begin{aligned} ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \\ + 2ux + 2vy + 2wz + d = 0. \quad . \quad . \quad . \quad . \quad (1) \end{aligned}$$

The line is given by the equations

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \qquad \text{(say)},$$

i.e. $$x = x_1 + rl,\ y = y_1 + rm,\ z = z_1 + rn. \quad . \quad . \quad . \quad (2)$$

Substituting from (2) in (1) for the points of intersection of the line and surface,

$$\begin{aligned} a(x_1 + rl)^2 + b(y_1 + rm)^2 + c(z_1 + rn)^2 + 2f(y_1 + rm)(z_1 + rn) \\ + 2g(z_1 + rn)(x_1 + rl) + 2h(x_1 + rl)(y_1 + rm) + 2u(x_1 + rl) \\ + 2v(y_1 + rm) + 2w(z_1 + rn) + d = 0, \end{aligned}$$

i.e. $r^2(al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm)$

$$+ r\begin{bmatrix} l(2ax_1 + 2hy_1 + 2gz_1 + 2u) + m(2hx_1 + 2by_1 + 2fz_1 + 2v) \\ + n(2gx_1 + 2fy_1 + 2cz_1 + 2w) \end{bmatrix}$$

$$+ ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1$$
$$+ 2ux_1 + 2vy_1 + 2wz_1 + d = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Writing

$$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$
$$+ 2ux + 2vy + 2wz + d,$$

$$\frac{\partial f}{\partial x} = 2ax + 2hy + 2gz + 2u, \quad \therefore \left(\frac{\partial f}{\partial x}\right)_1 = 2ax_1 + 2hy_1 + 2gz_1 + 2u;$$

$$\frac{\partial f}{\partial y} = 2hx + 2by + 2fz + 2v, \quad \therefore \left(\frac{\partial f}{\partial y}\right)_1 = 2hx_1 + 2by_1 + 2fz_1 + 2v;$$

$$\frac{\partial f}{\partial z} = 2gx + 2fy + 2cz + 2w, \quad \therefore \left(\frac{\partial f}{\partial z}\right)_1 = 2gx_1 + 2fy_1 + 2cz_1 + 2w;$$

and the equation (3) becomes

$$r^2(al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm)$$
$$+ r\left[l\left(\frac{\partial f}{\partial x}\right)_1 + m\left(\frac{\partial f}{\partial y}\right)_1 + n\left(\frac{\partial f}{\partial z}\right)_1\right] + f(x_1, y_1, z_1) = 0.$$

This result is very important and will be used in later theorems. From it can be seen that any straight line cuts the surface in two points, since the equation is a quadratic in r.

Hence any straight line in a plane will cut the surface in two points, and each plane section will therefore be a conic. Thus surfaces represented by the general equation of the second degree will have sections that are conics, and are therefore classed together under the name of *conicoids* (or *quadric surfaces*).

3. Theorem.—To find the equation of the tangent plane at the point (x_1, y_1, z_1) of the general conicoid $f(x, y, z) = 0$, where $f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d$.

From the previous theorem, the straight line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \quad . \quad . \quad . \quad . \quad (4)$$

cuts the conicoid where

$$r^2(al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm)$$
$$+ r\left[l\left(\frac{\partial f}{\partial x}\right)_1 + m\left(\frac{\partial f}{\partial y}\right)_1 + n\left(\frac{\partial f}{\partial z}\right)_1\right] + f(x_1, y_1, z_1) = 0. \quad (5)$$

If the point (x_1, y_1, z_1) lie on the surface,

$$f(x_1, y_1, z_1) = 0,$$

and one of the values of r in (5) is zero.

A second value of r will be zero in (5) if

$$l\left(\frac{\partial f}{\partial x}\right)_1 + m\left(\frac{\partial f}{\partial y}\right)_1 + n\left(\frac{\partial f}{\partial z}\right)_1 = 0. \quad . \quad . \quad . \quad . \quad (6)$$

Thus (6) is the condition that the straight line (4) shall be a tangent to the surface at (x_1, y_1, z_1).

Eliminating l, m, n between equations (4) and (6),

$$(x - x_1)\left(\frac{\partial f}{\partial x}\right)_1 + (y - y_1)\left(\frac{\partial f}{\partial y}\right)_1 + (z - z_1)\left(\frac{\partial f}{\partial z}\right)_1 = 0. \quad (7)$$

The equation (7) is the equation of a plane, being linear in x, y, and z, and all the tangent lines at the point (x_1, y_1, z_1) must lie in this plane.

Thus the equation (7) represents the tangent plane at the point (x_1, y_1, z_1) of the surface.

Writing the equation (7) in full, and cancelling by 2, it becomes

$$(x - x_1)(ax_1 + hy_1 + gz_1 + u) + (y - y_1)(hx_1 + by_1 + fz_1 + v)$$
$$+ (z - z_1)(gx_1 + fy_1 + cz_1 + w) = 0,$$

i.e. $axx_1 + byy_1 + czz_1 + f(yz_1 + zy_1) + g(zx_1 + xz_1) + h(xy_1 + yx_1)$
$+ ux + vy + wz$
$= ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 + ux_1 + vy_1 + wz_1.$

Adding $ux_1 + vy_1 + wz_1 + d$ to each side of this equation, and using the fact that $f(x_1, y_1, z_1) = 0$, since (x_1, y_1, z_1) lies on the surface, the equation of the tangent plane becomes

$$\boldsymbol{axx_1 + byy_1 + czz_1 + f(yz_1 + zy_1) + g(zx_1 + xz_1) + h(xy_1 + yx_1)}$$
$$\boldsymbol{+ u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0.}$$

This form of the equation of the tangent plane gives the following

rules for writing down the equation of the tangent plane at the point (x_1, y_1, z_1) of a conicoid: Replace

$$x^2 \text{ by } xx_1, \quad y^2 \text{ by } yy_1, \quad z^2 \text{ by } zz_1,$$
$$2yz \text{ by } (yz_1 + zy_1), \quad 2zx \text{ by } (zx_1 + xz_1), \quad 2xy \text{ by } (xy_1 + yx_1),$$
$$2x \text{ by } (x + x_1), \quad 2y \text{ by } (y + y_1), \quad 2z \text{ by } (z + z_1),$$

and leave the constant term unchanged.

Note.—The equation of the tangent plane can also be written in the form:

$$x(ax_1 + hy_1 + gz_1 + u) + y(hx_1 + by_1 + fz_1 + v) \\ + z(gx_1 + fy_1 + cz_1 + w) + ux_1 + vy_1 + wz_1 + d = 0.$$

If this plane pass through the point (α, β, γ), then

$$\alpha(ax_1 + hy_1 + gz_1 + u) + \beta(hx_1 + by_1 + fz_1 + v) \\ + \gamma(gx_1 + fy_1 + cz_1 + w) + ux_1 + vy_1 + wz_1 + d = 0.$$

This condition can be written as

$$x_1(a\alpha + h\beta + g\gamma + u) + y_1(h\alpha + b\beta + f\gamma + v) \\ + z_1(g\alpha + f\beta + c\gamma + w) + u\alpha + v\beta + w\gamma + d = 0, \quad . \quad (8)$$

a result that shows that all the tangent lines from (α, β, γ) to the surface touch the surface at points lying on the plane

$$x(a\alpha + h\beta + g\gamma + u) + y(h\alpha + b\beta + f\gamma + v) \\ + z(g\alpha + f\beta + c\gamma + w) + u\alpha + v\beta + w\gamma + d = 0.$$

These tangent lines from (α, β, γ) to the conicoid form what is known as a *cone* (not necessarily circular). The plane of the points of contact of the tangent lines is known as the *polar plane* of the point (α, β, γ) with respect to the conicoid, and the point (α, β, γ) is the *pole* of this plane.

From its formation it can be seen that the polar plane of the point (x_1, y_1, z_1) with respect to the surface can be written down by the same rule as for the tangent plane at the point (x_1, y_1, z_1). But it must be remembered that, in the case of the polar plane, the point (x_1, y_1, z_1) does not generally lie on the surface.

From the symmetry of the equation (8) in (α, β, γ) and (x_1, y_1, z_1), which can be interchanged without altering the equation, it follows that, if the polar plane of P passes through Q, then the polar plane of Q passes through P.

4. Theorem.—To find the condition that the plane

$$lx + my + nz = p$$

shall touch the surface

$$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$$

The equation of the tangent plane at the point (x_1, y_1, z_1) can be written in the form

$$x(ax_1 + hy_1 + gz_1 + u) + y(hx_1 + by_1 + fz_1 + v) + z(gx_1 + fy_1 + cz_1 + w) + ux_1 + vy_1 + wz_1 + d = 0. \quad (9)$$

If this be equivalent to the plane given by

$$lx + my + nz = p, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

comparing coefficients,

$$\frac{ax_1 + hy_1 + gz_1 + u}{l} = \frac{hx_1 + by_1 + fz_1 + v}{m} = \frac{gx_1 + fy_1 + cz_1 + w}{n}$$

$$= \frac{ux_1 + vy_1 + wz_1 + d}{-p} = -k \qquad \text{(say)},$$

$$\therefore\ ax_1 + hy_1 + gz_1 + u + kl = 0, \quad . \quad . \quad . \quad . \quad (11)$$

$$hx_1 + by_1 + fz_1 + v + km = 0, \quad . \quad . \quad . \quad . \quad (12)$$

$$gx_1 + fy_1 + cz_1 + w + kn = 0, \quad . \quad . \quad . \quad . \quad (13)$$

$$ux_1 + vy_1 + wz_1 + d - kp = 0. \quad . \quad . \quad . \quad . \quad (14)$$

Also, since (x_1, y_1, z_1) lies on the plane (9),

$$lx_1 + my_1 + nz_1 - p = 0. \quad . \quad . \quad . \quad . \quad (15)$$

Eliminating k, x_1, y_1, z_1 between the equations (11), (12), (13), (14), and (15), the required condition is

$$\begin{vmatrix} a & h & g & u & l \\ h & b & f & v & m \\ g & f & c & w & n \\ u & v & w & d & -p \\ l & m & n & -p & 0 \end{vmatrix} = 0.$$

Example 1 (L.U.).—If p be the length of the perpendicular from the centre of the ellipsoid $ax^2 + by^2 + cz^2 = 1$ upon the tangent plane at the point (α, β, γ), and if l be the length of the normal at (α, β, γ) intercepted by the ellipsoid, prove that

$$2/(p^3l) = a^3\alpha^2 + b^3\beta^2 + c^3\gamma^2.$$

The equation of the tangent plane at (α, β, γ) to the surface

$$ax^2 + by^2 + cz^2 = 1 \quad \ldots \ldots \ldots \quad \text{(i)}$$

is

$$a\alpha x + b\beta y + c\gamma z = 1 \quad \ldots \ldots \ldots \quad \text{(ii)}$$

(using the rule for writing down the tangent plane).

The centre of the ellipsoid (i) is readily seen to be (0, 0, 0) by symmetry.

Hence
$$p = \frac{1}{\sqrt{(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)}}.$$

Since the normal at (α, β, γ) is perpendicular to the tangent plane (ii), its equation is

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{c\gamma} = r \quad \text{(say)},$$

$$\therefore\ x = \alpha(1 + ar), \quad y = \beta(1 + br), \quad z = \gamma(1 + cr). \quad \ldots \quad \text{(iii)}$$

Substituting from (iii) in (i) for the points of intersection of the normal and ellipsoid,

$$a\alpha^2(1 + ar)^2 + b\beta^2(1 + br)^2 + c\gamma^2(1 + cr)^2 = 1,$$

i.e

$$(a\alpha^2 + b\beta^2 + c\gamma^2) + 2r(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2) + r^2(a^3\alpha^2 + b^3\beta^2 + c^3\gamma^2) = 1. \quad . \quad \text{(iv)}$$

But (α, β, γ) lies on the surface (i),

$$\therefore\ a\alpha^2 + b\beta^2 + c\gamma^2 = 1,$$

and, therefore, equation (iv) becomes

$$r^2(a^3\alpha^2 + b^3\beta^2 + c^3\gamma^2) + 2r(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2) = 0.$$

The value $r = 0$ from this equation gives the point (α, β, γ), and the other point is given by

$$r = -\frac{2(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)}{a^3\alpha^2 + b^3\beta^2 + c^3\gamma^2}. \quad \ldots \ldots \ldots \quad \text{(v)}$$

But $a\alpha$, $b\beta$, $c\gamma$ are the proportional direction cosines of the normal,

$$\therefore\ l = r\sqrt{(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)}$$

$$= \frac{2(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)^{3/2}}{a^3\alpha^2 + b^3\beta^3 + c^3\gamma^2} \quad \text{[using (v) and neglecting sign]}$$

$$= \frac{2}{p^3(a^3\alpha^2 + b^3\beta^2 + c^3\gamma^2)}.$$

$$\therefore\ \frac{2}{p^3 l} = a^3\alpha^2 + b^3\beta^2 + c^3\gamma^2.$$

5. Theorem.—To find the equation of the plane containing the chords of a conicoid bisected at a given point (x_1, y_1, z_1).

With the previous notation, let $f(x, y, z) = 0$ represent the conicoid.

The points where this is cut by the line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \qquad \text{(say)}$$

are given by (Section 2)

$$r^2(al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm) + r\left[l\left(\frac{\partial f}{\partial x}\right)_1 + m\left(\frac{\partial f}{\partial y}\right)_1 + n\left(\frac{\partial f}{\partial z}\right)_1\right] + f(x_1, y_1, z_1) = 0. \quad (16)$$

Now (x_1, y_1, z_1) will be the mid-point of the chord if the roots of equation (16), considered as a quadratic in r, are equal and opposite. The condition for this is

$$l\left(\frac{\partial f}{\partial x}\right)_1 + m\left(\frac{\partial f}{\partial y}\right)_1 + n\left(\frac{\partial f}{\partial z}\right)_1 = 0. \quad . \quad . \quad . \quad (17)$$

This equation shows that there are an infinite number of chords bisected at the point (x_1, y_1, z_1), and combining the equation (17) with the original equation of the chord, it can be seen that all these chords lie in the plane represented by the equation

$$(x - x_1)\left(\frac{\partial f}{\partial x}\right)_1 + (y - y_1)\left(\frac{\partial f}{\partial y}\right)_1 + (z - z_1)\left(\frac{\partial f}{\partial z}\right)_1 = 0.$$

Note.—This equation is the same as for the tangent plane at the point (x_1, y_1, z_1) of the conicoid, but it must be remembered in this case that the point (x_1, y_1, z_1) does *not* lie on the conicoid.

N.B.—The centre of a conic is defined as being the point which bisects all chords of the conic passing through it. Hence in order to find the centre of the conic formed by the plane $lx + my + nz = p$ cutting a given conicoid, the following procedure is adopted:

Let (x_1, y_1, z_1) be the centre of the conic formed by the plane and the conicoid. The equation of the plane will then be

$$(x - x_1)\left(\frac{\partial f}{\partial x}\right)_1 + (y - y_1)\left(\frac{\partial f}{\partial y}\right)_1 + (z - z_1)\left(\frac{\partial f}{\partial z}\right)_1 = 0.$$

Comparing this with the equation $lx + my + nz = p$, to which it must be equivalent,

$$\frac{\left(\frac{\partial f}{\partial x}\right)_1}{l} = \frac{\left(\frac{\partial f}{\partial y}\right)_1}{m} = \frac{\left(\frac{\partial f}{\partial z}\right)_1}{n} = \frac{x_1\left(\frac{\partial f}{\partial x}\right)_1 + y_1\left(\frac{\partial f}{\partial y}\right)_1 + z_1\left(\frac{\partial f}{\partial z}\right)_1}{p},$$

and from these three equations the values of x_1, y_1, z_1 can be determined. (An example on this will be given under " The Ellipsoid ".)

6. Theorem.—To find the locus of the mid-points of parallel chords of the conicoid $f(x, y, z) = 0$ [with the usual notation for $f(x, y, z)$].

Consider the line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r.$$

If, when forming a chord of the conicoid $f(x, y, z) = 0$, this line be bisected at the point (x_1, y_1, z_1), then, as in the previous theorem,

$$l\left(\frac{\partial f}{\partial x}\right)_1 + m\left(\frac{\partial f}{\partial y}\right)_1 + n\left(\frac{\partial f}{\partial z}\right)_1 = 0.$$

But l, m, n are constants for parallel chords, and (x_1, y_1, z_1) is any point on the required locus. Hence the equation of the required locus is

$$l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} + n\frac{\partial f}{\partial z} = 0.$$

On expansion it is seen that this equation is linear in x, y, and z, and therefore represents a plane, which is known as the *diametral plane* of the system of parallel chords of the conicoid. When this diametral plane is perpendicular to the chords it bisects, it is known as a *principal plane* of the conicoid.

The equation of the diametral plane bisecting all chords having direction cosines proportional to l, m, n is given by

$$l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} + n\frac{\partial f}{\partial z} = 0,$$

which, after division by 2 and expansion, reduces to

$$l(ax + hy + gz + u) + m(hx + by + fz + v) \\ + n(gx + fy + cz + w) = 0,$$

i.e. $$x(al + hm + gn) + y(hl + bm + fn) \\ + z(gl + fm + cn) + (ul + vm + wn) = 0.$$

If this plane be perpendicular to the chords that it bisects, then

$$\frac{al + hm + gn}{l} = \frac{hl + bm + fn}{m} = \frac{gl + fm + cn}{n} = \lambda \quad \text{(say)},$$

i.e.
$$l(a - \lambda) + mh + ng = 0, \quad \ldots \ldots \quad (18)$$
$$lh + m(b - \lambda) + nf = 0, \quad \ldots \ldots \quad (19)$$
$$lg + mf + n(c - \lambda) = 0. \quad \ldots \ldots \quad (20)$$

Eliminating l, m, n between equations (18), (19), and (20),

$$\begin{vmatrix} \lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0. \quad . \quad . \quad . \quad (21)$$

The equation (21) is a cubic in λ and gives three values (real, coincident, or imaginary) of λ from which can be obtained the equations of the principal planes. One of the roots of any cubic must be real, and hence there must be *at least one real principal plane to any conicoid.*

7. Definition.—The centre of a conicoid is the point bisecting all chords of the conicoid passing through it.

Thus, if the centre of a conicoid be taken as the origin, and the point (x_1, y_1, z_1) lies on the conicoid, it follows that the point $(-x_1, -y_1, -z_1)$ must lie on the conicoid. Using this fact, it can be shown that, if the centre of a conicoid be taken as the origin, there will be no linear terms in x, y, or z present in the equation of the conicoid. Hence the equation of any central conicoid having its centre at the origin will have as its equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d = 0.$$

N.B.—Not all quadric surfaces have a centre. An example of this is the paraboloid $ax^2 + by^2 = cz$, given later.

THE SPHERE

8. The equation of the sphere, centre (x_1, y_1, z_1), radius r, has been shown in the previous chapter to be

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2.$$

In the expanded form, this equation is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

This latter equation is the general equation of the sphere and, since it contains four independent constants, a sphere can be made to satisfy four independent conditions, such as passing through four given points, touching four given planes, etc.

It is to be noted that the equation of a sphere contains no terms in yz, zx, xy, and the coefficients of x^2, y^2, and z^2 are equal.

9. Theorem.—To find, from first principles, the equation of the tangent plane at the point (x_1, y_1, z_1) of the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

The equation of the sphere can be written

$$(x + u)^2 + (y + v)^2 + (z + w)^2 = u^2 + v^2 + w^2 - d,$$

which shows that its centre is $(-u, -v, -w)$ and its radius

$$= \sqrt{(u^2 + v^2 + w^2 - d)}.$$

The radius joining the centre of the sphere to the point (x_1, y_1, z_1) has direction cosines proportional to $(x_1 + u)$, $(y_1 + v)$, $(z_1 + w)$, and therefore the equation of the tangent plane at (x_1, y_1, z_1), which will be perpendicular to this radius by geometry, is

$$(x_1 + u)(x - x_1) + (y_1 + v)(y - y_1) + (z_1 + w)(z - z_1) = 0,$$

i.e.

$$xx_1 + yy_1 + zz_1 + ux + vy + wz = x_1{}^2 + y_1{}^2 + z_1{}^2 + ux_1 + vy_1 + wz_1.$$

Adding the quantity $ux_1 + vy_1 + wz_1 + d$ to each side, the equation becomes

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d$$
$$= x_1{}^2 + y_1{}^2 + z_1{}^2 + 2ux_1 + 2vy_1 + 2wz_1 + d.$$

The right-hand side of this equation is zero, since (x_1, y_1, z_1) lies on the sphere, and hence the equation of the tangent plane is

$$\boldsymbol{xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0.}$$

Corollary.—In the case of the sphere $x^2 + y^2 + z^2 = r^2$, whose centre is O and radius r, the equation of the tangent plane at the point (x_1, y_1, z_1) is

$$\boldsymbol{xx_1 + yy_1 + zz_1 = r^2.}$$

10. Theorem.—To find the condition that the plane

$$lx + my + nz = p$$

shall touch the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

The centre of the given sphere is $(-u, -v, -w)$, and its radius is $\sqrt{(u^2 + v^2 + w^2 - d)}$.

The perpendicular on the plane from the centre of the sphere

$$= \pm \frac{p + lu + mv + nw}{\sqrt{(l^2 + m^2 + n^2)}}$$

$$= \text{radius of sphere}.$$

Hence the required condition is

$$\pm \frac{p + lu + mv + nw}{\sqrt{(l^2 + m^2 + n^2)}} = \sqrt{(u^2 + v^2 + w^2 - d)},$$

i.e. $(p + lu + mv + nw)^2 = (u^2 + v^2 + w^2 - d)(l^2 + m^2 + n^2).$

In the case of the sphere $x^2 + y^2 + z^2 = r^2$, the condition becomes

$$p^2 = r^2(l^2 + m^2 + n^2).$$

11. Theorem.—The polar plane of the point (x_1, y_1, z_1), with respect to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, is

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0.$$

This result can be proved by the same process as in the case of the general conicoid.

12. Theorem.—If the plane $lx + my + nz = p$ be a tangent plane to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, to find its point of contact.

Let the point of contact be (x_1, y_1, z_1). The equation of the tangent plane at this point to the sphere will be

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0.$$

Comparing this with the given equation of the tangent plane,

$$\frac{x_1 + u}{l} = \frac{y_1 + v}{m} = \frac{z_1 + w}{n} = \frac{ux_1 + vy_1 + wz_1 + d}{-p}.$$

On solving these three equations in x_1, y_1, z_1, the values of x_1, y_1, z_1 can be found.

Note.—A similar method can be adopted in order to find the pole of a given plane with respect to a given sphere.

13. Theorem.—To find the length of the tangent from the point (x_1, y_1, z_1) to the sphere $f(x, y, z) = 0$, where

$$f(x, y, z) \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d.$$

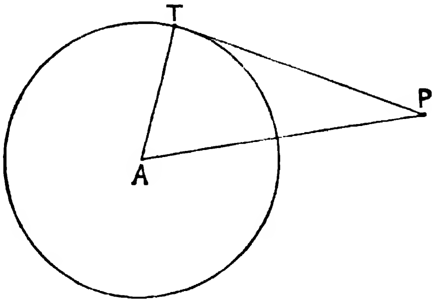
Let P be the point (x_1, y_1, z_1), and A the centre of the sphere. Then

$$A \equiv (-u, -v, -w).$$

T is the point of contact of a tangent from P to the sphere. Then PT is perpendicular to AT, and since AT is the radius,

$$AT^2 = u^2 + v^2 + w^2 - d.$$

Now $$AP^2 = (x_1 + u)^2 + (y_1 + v)^2 + (z_1 + w)^2.$$



By Pythagoras' theorem,

$$PT^2 = AP^2 - AT^2,$$

i.e. $$\begin{aligned} PT^2 &= (x_1 + u)^2 + (y_1 + v)^2 + (z_1 + w)^2 - (u^2 + v^2 + w^2 - d) \\ &= x_1{}^2 + y_1{}^2 + z_1{}^2 + 2ux_1 + 2vy_1 + 2wz_1 + d \\ &= f(x_1, y_1, z_1). \end{aligned}$$

N.B.—Before this result can be used the coefficients of x^2, y^2 and z^2 in the equation of the sphere must be reduced to unity if necessary.

14. Theorem.—To find the condition that the two spheres

$$x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$

and $$x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

shall be orthogonal (i.e. cut at right angles).

As in the case of orthogonal circles, if A and B be the centres of the two spheres, then the condition that the two spheres shall be orthogonal is that the lines (radii) joining A and B to a point on the circle of intersection of the two spheres shall be perpendicular.

If P be any point on the intersection of the two spheres, then, for the two spheres to be orthogonal,

$$AP^2 + BP^2 = AB^2.$$

But $$A \equiv (-u_1, -v_1, -w_1); \; B \equiv (-u_2, -v_2, -w_2);$$

$$AP^2 = u_1{}^2 + v_1{}^2 + w_1{}^2 - d_1; \; BP^2 = u_2{}^2 + v_2{}^2 + w_2{}^2 - d_2;$$

and $$AB^2 = (u_1 - u_2)^2 + (v_1 - v_2)^2 + (w_1 - w_2)^2.$$

Hence the required condition is

$$(u_1^2 + v_1^2 + w_1^2 - d_1) + (u_2^2 + v_2^2 + w_2^2 - d_2)$$
$$= (u_1 - u_2)^2 + (v_1 - v_2)^2 + (w_1 - w_2)^2,$$

i.e.
$$\mathbf{2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2.}$$

N.B.—As in the case of two circles, if $S_1 = 0$ and $S_2 = 0$ represent two spheres, the equation of any sphere passing through their circle of intersection is given by $\boldsymbol{S_1 + \lambda S_2 = 0}$, where λ is any arbitrary constant. If $\lambda = -1$, this sphere degenerates into the plane of intersection known as the *radical plane* of the coaxal system $S_1 = 0$, $S_2 = 0$, providing that the coefficients of x^2, y^2, z^2 are the same in the equations of both spheres.

Also, if $P = 0$ be the equation of any plane, the equation of any sphere passing through the circle of intersection of this plane and the sphere $S = 0$ is given by

$$\boldsymbol{S + \lambda P = 0},$$

where λ is any arbitrary constant, and the plane $P = 0$ is the *radical plane* of the coaxal system.

Example 2 (L.U.).—Through the circle of intersection of the sphere

$$x^2 + y^2 + z^2 = 25,$$

and the plane $x + 2y + 2z - 9 = 0$, two spheres S_1 and S_2 are drawn to touch the plane $4x + 3y - 30 = 0$. Find the equations of S_1 and S_2, and the co-ordinates of the point through which pass all common tangent planes of S_1 and S_2.

Any sphere through the circle of intersection of the sphere

$$x^2 + y^2 + z^2 = 25, \quad \ldots \ldots \quad \text{(i)}$$

and the plane
$$x + 2y + 2z - 9 = 0, \quad \ldots \ldots \quad \text{(ii)}$$

is given by
$$x^2 + y^2 + z^2 - 25 + \lambda(x + 2y + 2z - 9) = 0. \quad \ldots \quad \text{(iii)}$$

By inspection, the centre of this sphere is $(-\frac{1}{2}\lambda, -\lambda, -\lambda)$, and its radius

$$\sqrt{(\tfrac{1}{4}\lambda^2 + \lambda^2 + \lambda^2 + 25 + 9\lambda)}.$$

The perpendicular from $(-\frac{1}{2}\lambda, -\lambda, -\lambda)$ on the plane

$$4x + 3y - 30 = 0, \quad \ldots \ldots \quad \text{(iv)}$$

is
$$\pm\frac{2\lambda + 3\lambda + 30}{\sqrt{(4^2 + 3^2)}} = \pm\frac{5\lambda + 30}{5} = \pm(\lambda + 6).$$

Hence, if the sphere (iii) is to touch the plane (iv), this perpendicular distance must equal the radius of the sphere,

i.e. $$\pm(\lambda+6)=\sqrt{\{(9/4)\lambda^2+9\lambda+25\}},$$

i.e. $$\lambda^2+12\lambda+36=9\lambda^2/4+9\lambda+25,$$

i.e. $$5\lambda^2-12\lambda-44=0,$$

i.e. $$(5\lambda-22)(\lambda+2)=0,$$

$$\therefore\ \lambda=22/5 \text{ or } -2.$$

For $\lambda=22/5$, the required sphere S_1 is given by

$$x^2+y^2+z^2-25+(22/5)(x+2y+2z-9)=0,$$

i.e. $$5x^2+5y^2+5z^2+22x+44y+44z=323.$$

For $\lambda=-2$, the required sphere S_2 is given by

$$x^2+y^2+z^2-25-2(x+2y+2z-9)=0,$$

i.e. $$x^2+y^2+z^2-2x-4y-4z=7.$$

Now the plane (iv) is a common tangent plane to the spheres S_1 and S_2, and therefore all common tangent planes of S_1 and S_2 will pass through the point of intersection of the line of centres of S_1 and S_2 and the plane (iv).

The centre of sphere S_1 is $(-\frac{11}{5}, -\frac{22}{5}, -\frac{22}{5})$, and the centre of sphere S_2 is $(1, 2, 2)$, therefore the equation of the line of centres is

$$\frac{x-1}{1+\frac{11}{5}}=\frac{y-2}{2+\frac{22}{5}}=\frac{z-2}{2+\frac{22}{5}},$$

i.e. $$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-2}{2}=r \qquad \text{(say).}$$

$$\therefore\ x=1+r,\quad y=2+2r,\quad z=2+2r.$$

Using this in (iv) for the required point,

$$(4+4r)+(6+6r)-30=0,$$

i.e. $$10r=20,\quad \therefore\ r=2,$$

and the required point is (3, 6, 6).

Example 3 (L.U.).—Spheres are drawn to pass through the points (2, 0, 0), (8, 0, 0) and to touch the axes of y and z. Find how many such spheres can be drawn, and give their equations. Show that the polar planes of the origin with respect to these spheres all pass through the same point, and find the co-ordinates of this point.

Let the equation of a required sphere be

$$x^2+y^2+z^2+2ux+2vy+2wy+d=0. \quad \ldots \quad \text{(i)}$$

Since it passes through (2, 0, 0) and (8, 0, 0),

$$4+4u+d=0, \quad \ldots \quad \text{(ii)}$$

$$64+16u+d=0. \quad \ldots \quad \text{(iii)}$$

(iii) — (ii) gives $$60+12u=0,\quad \therefore\ u=-5.$$

Therefore, from (ii), $$d=16.$$

The axis of y is $x=0$, $z=0$, and where this cuts the sphere (i),

$$y^2+2vy+d=0. \quad \ldots \quad \text{(iv)}$$

Since the y-axis is a tangent to the sphere (i), the roots of (iv) in y are coincident, i.e.

$$v^2 = d = 16, \quad \therefore v = \pm 4.$$

The axis of z is $x = 0$, $y = 0$, and where this cuts the sphere (i),

$$z^2 + 2wz + d = 0,$$

and as above

$$w = \pm 4.$$

Hence four spheres can fulfil the given conditions, and the equations of these are:

$$\left.\begin{aligned} x^2 + y^2 + z^2 - 10x + 8y + 8z + 16 &= 0,\\ x^2 + y^2 + z^2 - 10x - 8y + 8z + 16 &= 0,\\ x^2 + y^2 + z^2 - 10x + 8y - 8z + 16 &= 0,\\ x^2 + y^2 + z^2 - 10x - 8y - 8z + 16 &= 0.\end{aligned}\right\}$$

The polar plane of (x_1, y_1, z_1) with respect to the sphere (i) is

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0,$$

and for the point (0, 0, 0) this plane becomes

$$ux + vy + wz + d = 0.$$

Hence the polar planes of the origin with respect to the above four spheres are respectively

$$-5x + 4y + 4z + 16 = 0, \qquad \text{(v)}$$
$$-5x - 4y + 4z + 16 = 0, \qquad \text{(vi)}$$
$$-5x + 4y - 4z + 16 = 0, \qquad \text{(vii)}$$
$$-5x - 4y - 4z + 16 = 0. \qquad \text{(viii)}$$

Solving (v), (vi), and (vii), $x = \frac{16}{5}$, $y = 0$, $z = 0$,

and this point also satisfies equation (viii). Hence the polar planes all pass through the same point $(\frac{16}{5}, 0, 0)$.

Example 4 (L.U.)—Prove that the circles

$$x^2 + y^2 + z^2 - 9x + 4y + 5z - 1 = 0,\ 7x - 2y + z = 4$$

and

$$x^2 + y^2 + z^2 + 6x - 10y + 6z - 7 = 0,\ 4x - 6y - 1 = 0$$

lie on a sphere. Find the equation of the sphere and the radii of the circles.

If λ_1 and λ_2 be arbitrary constants, then the equations of any spheres passing through the two circles will be given respectively by

$$x^2 + y^2 + z^2 - 9x + 4y + 5z - 1 + \lambda_1(7x - 2y + z - 4) = 0, \qquad \text{(i)}$$
$$x^2 + y^2 + z^2 + 6x - 10y + 6z - 7 + \lambda_2(4x - 6y - 1) = 0. \qquad \text{(ii)}$$

If the two circles are to lie on a sphere, then, for some values of λ_1 and λ_2, the equations (i) and (ii) will be identical.

Equating coefficients of x, y, unity, and z in the two equations,

$$-9 + 7\lambda_1 = 6 + 4\lambda_2, \quad \text{i.e. } 7\lambda_1 - 4\lambda_2 - 15 = 0. \qquad \text{(iii)}$$
$$4 - 2\lambda_1 = -10 - 6\lambda_2, \quad \text{i.e. } \lambda_1 - 3\lambda_2 - 7 = 0. \qquad \text{(iv)}$$
$$-1 - 4\lambda_1 = -7 - \lambda_2, \quad \text{i.e. } 4\lambda_1 - \lambda_2 - 6 = 0. \qquad \text{(v)}$$
$$5 + \lambda_1 = 6, \quad \text{i.e. } \lambda_1 = 1. \qquad \text{(vi)}$$

Solving (iii) and (iv) by determinants,

$$\frac{\lambda_1}{+28-45}=\frac{-\lambda_2}{-49+15}=\frac{1}{-21+4}.$$

$$\therefore\ \frac{\lambda_1}{-17}=\frac{\lambda_2}{34}=\frac{1}{-17},\quad \therefore\ \lambda_1=1,\ \lambda_2=-2.$$

Hence (vi) is satisfied, and using these values in (v),

$$4+2-6=0,\quad \text{i.e. } 0=0.$$

Hence unique values of λ_1 and λ_2 can be found so that equations (i) and (ii) represent the same sphere, and therefore the two given circles must lie on the same sphere.

Using $\lambda_1 = 1$, the equation of the sphere is

$$x^2+y^2+z^2-9x+4y+5z-1+(7x-2y+z-4)=0,$$

i.e.
$$x^2+y^2+z^2-2x+2y+6z-5=0.$$

The centre of this sphere is $(1, -1, -3)$ and its radius is

$$\sqrt{\{1^2+(-1)^2+(-3)^2-(-5)\}}=\sqrt{16}=4.$$

Let r_1 and r_2 be the radii of the two circles respectively, then $r_1=\sqrt{(r^2-p_1^2)}$, $r_2=\sqrt{(r^2-p_2^2)}$, where r is the radius of the sphere on which the two circles lie, and p_1 and p_2 are the perpendicular distances of its centre from the planes $7x-2y+z-4=0$ and $4x-6y-1=0$ containing the circles.

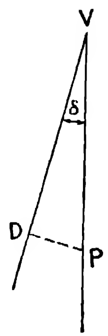
Now
$$r=4;$$

$$p_1=\frac{2}{\sqrt{(7^2+2^2+1^2)}}=\frac{2}{\sqrt{54}};\quad p_2=\frac{9}{\sqrt{(4^2+6^2)}}=\frac{9}{\sqrt{52}}.$$

$$\therefore\ r_1=\sqrt{(16-\tfrac{4}{54})}=2\sqrt{\tfrac{215}{54}};\quad r_2=\sqrt{(16-\tfrac{81}{52})}=\sqrt{\tfrac{751}{52}}.$$

The Right Circular Cone and Cylinder

15. If the axis $\dfrac{x-a}{l}=\dfrac{y-b}{m}=\dfrac{z-c}{n}$, vertex $V\equiv(a, b, c)$, and semi-vertical angle δ of a cone be known, the equation of the cone is obtained as follows:



Let $P\equiv(x, y, z)$ be any point on the cone, and PD the perpendicular from P on the axis.

It was proved in the previous chapter that

$$PD^2=(x-a)^2+(y-b)^2+(z-c)^2-\left\{\frac{l(x-a)+m(y-b)+n(z-c)}{\sqrt{(l^2+m^2+n^2)}}\right\}^2.$$

But
$$PV^2=(x-a)^2+(y-b)^2+(z-c)^2.$$

From the diagram, $PD = PV\sin\delta$,

$$\therefore\ PD^2=PV^2\sin^2\delta.$$

Using this, the required equation of the cone is

$$(x-a)^2+(y-b)^2+(z-c)^2-\left\{\frac{l(x-a)+m(y-b)+n(z-c)}{\sqrt{(l^2+m^2+n^2)}}\right\}^2$$
$$=\left\{(x-a)^2+(y-b)^2+(z-c)^2\right\}\sin^2\delta,$$

i.e. $(l^2+m^2+n^2)\cos^2\delta\left\{(x-a)^2+(y-b)^2+(z-c)^2\right\}$
$$=\left\{l(x-a)+m(y-b)+n(z-c)\right\}^2.$$

Using the same axis for a right circular cylinder of radius r, it is seen that $PD^2=r^2$ and the equation of the circular cylinder will be

$$(x-a)^2+(y-b)^2+(z-c)^2-\frac{\{l(x-a)+m(y-b)+n(z-c)\}^2}{l^2+m^2+n^2}=r^2.$$

Example 5 (L.U.).—Find the perpendicular distance of the point (f, g, h) from the line $\dfrac{x-a}{l}=\dfrac{y-b}{m}=\dfrac{z-c}{n}$.

Find the equation of the right circular cone, whose vertex is at the origin, whose axis is the line $\dfrac{x}{1}=\dfrac{y}{2}=\dfrac{z}{3}$, and which has a semi-vertical angle of 30°. Show that the sections of this cone by planes parallel to $x=0$ are hyperbolas, and by planes parallel to $y=0$ and $z=0$ are ellipses.

The first part of the question is bookwork, and using the previous result for the cone, with $l=1$, $m=2$, $n=3$; $a=b=c=0$; and $\delta=30°$, its equation is

$$(1^2+2^2+3^2)\tfrac{3}{4}\{x^2+y^2+z^2\}=(x+2y+3z)^2,$$

i.e. $$42(x^2+y^2+z^2)=4(x^2+4y^2+9z^2+4xy+6xz+12yz),$$

i.e. $$38x^2+26y^2+6z^2-16xy-24xz-48yz=0,$$

i.e. $$19x^2+13y^2+3z^2-24yz-12zx-8xy=0. \quad . \quad . \quad . \quad \text{(i)}$$

Taking a plane $x=c_1$ parallel to $x=0$, its intersection with the cone is

$$19c_1^2+13y^2+3z^2-8c_1y-12c_1z-24yz=0,$$

i.e. $$13y^2-24yz+3z^2-8c_1y-12c_1z+19c_1^2=0.$$

Since $\qquad 12^2>13\times 3 \qquad (h^2>ab$ in general equation),

this curve is a hyperbola.

Taking a plane $y=c_2$ parallel to $y=0$, its intersection with the cone is given by

$$19x^2-12zx+3z^2-8c_2x-24c_2z+13c_2^2=0.$$

Since $\qquad 6^2<19\times 3 \qquad (h^2<ab),$

the curve is an ellipse.

Taking the plane $z = c_3$ parallel to $z = 0$, its intersection with the cone is given by

$$19x^2 - 8xy + 13y^2 - 12c_3x - 24c_3y + 3c_3^2 = 0.$$

Since $$4^2 < 13 \times 19 \qquad (h^2 < ab),$$

the curve is an ellipse.

THE ELLIPSOID

16. The standard equation of an ellipsoid whose centre is at the origin, and whose axes are the axes of co-ordinates, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

If $a \neq b \neq c$, it can be shown that all sections of the ellipsoid are ellipses. Hence the name *ellipsoid*.

17. Theorem.—To find, from first principles, the equation of the tangent plane at the point (x_1, y_1, z_1) of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Consider the line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \quad \text{(say)}, \qquad (22)$$

i.e. $$x = x_1 + rl, \; y = y_1 + rm, \; z = z_1 + rn. \qquad (23)$$

Substituting from (23) in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \ldots\ldots \quad (24)$$

for the points of intersection of the line and the ellipsoid,

$$\frac{(x_1 + rl)^2}{a^2} + \frac{(y_1 + rm)^2}{b^2} + \frac{(z_1 + rn)^2}{c^2} = 1,$$

i.e. $$r^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) + 2r\left(\frac{lx_1}{a^2} + \frac{my_1}{b^2} + \frac{nz_1}{c^2}\right)$$

$$+ \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1. \qquad (25)$$

If (x_1, y_1, z_1) lie on the ellipsoid, the equation (25) reduces to

$$r^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) + 2r\left(\frac{lx_1}{x^2} + \frac{my_1}{b^2} + \frac{nz_1}{c^2}\right) = 0. \quad (26)$$

One value of r in (26) is zero, and a second root will be zero if

$$\frac{lx_1}{a^2} + \frac{my_1}{b^2} + \frac{nz_1}{c^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Hence equation (27) will be the condition that the line (22) is a tangent line to the ellipsoid (24) at (x_1, y_1, z_1).

Eliminating (l, m, n) between equations (22) and (27), all tangent lines at (x_1, y_1, z_1) will lie on the plane

$$\frac{x_1(x - x_1)}{a^2} + \frac{y_1(y - y_1)}{b^2} + \frac{z_1(z - z_1)}{c^2} = 0,$$

i.e.
$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2}.$$

But (x_1, y_1, z_1) lies on the ellipsoid (24),

$$\therefore \ \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1,$$

and the equation of the tangent plane at (x_1, y_1, z_1) becomes

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1.$$

Using this equation, the normal at (x_1, y_1, z_1) will have direction cosines proportional to x_1/a^2, y_1/b^2, z_1/c^2. Therefore the equation of the normal at (x_1, y_1, z_1) will be

$$\frac{x - x_1}{x_1/a^2} = \frac{y - y_1}{y_1/b^2} = \frac{z - z_1}{z_1/c^2}.$$

As proved in the general case of a conicoid, the equation of the plane containing all chords of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, bisected at the point (α, β, γ), will be given by

$$\frac{x\alpha}{a^2} + \frac{y\beta}{b^2} + \frac{z\gamma}{c^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}.$$

Also the equation of the diametral plane bisecting all chords having direction cosines l, m, n will be

$$\frac{lx}{a^2} + \frac{my}{b^2} + \frac{nz}{c^2} = 0.$$

Example 6 (L.U.).—Find the equation of the plane that cuts the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

in a conic of which the point (α, β, γ) is the centre.

Find the co-ordinates of the centre of either of the two conics in which the ellipsoid $x^2+2y^2+3z^2=1$ is cut by a plane parallel to the plane $y-2z=0$ and half-way between this plane and a parallel tangent plane to the ellipsoid.

Part (i).—The given ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \quad \ldots \quad \text{(i)}$$

Consider the line $$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \quad \text{(say).} \quad \ldots \quad \text{(ii)}$$

Then $$x=\alpha+rl,\quad y=\beta+rm,\quad z=\gamma+rn. \quad \ldots \quad \text{(iii)}$$

Substituting from (iii) in (i) for the points of intersection of the line (ii) and ellipsoid (i),

$$\frac{(\alpha+rl)^2}{a^2}+\frac{(\beta+rm)^2}{b^2}+\frac{(\gamma+rn)^2}{c^2}=1,$$

i.e. $$r^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)+2r\left(\frac{\alpha l}{a^2}+\frac{\beta m}{b^2}+\frac{\gamma n}{c^2}\right)+\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=1. \quad \text{(iv)}$$

If (α, β, γ) be the mid-point of the chord, then the roots of (iv), considered as a quadratic in r, must be equal and opposite, and the condition for this is

$$\frac{\alpha l}{a^2}+\frac{\beta m}{b^2}+\frac{\gamma n}{c^2}=0. \quad \ldots \quad \text{(v)}$$

Eliminating l, m, n between equations (ii) and (v), all chords of the ellipsoid (i) bisected at (α, β, γ) lie in the plane

$$\frac{\alpha(x-\alpha)}{a^2}+\frac{\beta(y-\beta)}{b^2}+\frac{\gamma(z-\gamma)}{c^2}=0,$$

i.e. $$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}.$$

Hence this is the required equation of the plane, since the point (α, β, γ) is the centre of the conic formed by this plane section, due to the fact that it bisects all chords passing through it.

Part (ii).—The equation of the ellipsoid is

$$x^2+2y^2+3z^2=1. \quad \ldots \quad \text{(i)}$$

The tangent plane at (x_1, y_1, z_1) is

$$xx_1+2yy_1+3zz_1=1. \quad \ldots \quad \text{(ii)}$$

If this plane be parallel to the plane

$$y - 2z = 0,$$

then
$$\frac{x_1}{0} = \frac{2y_1}{1} = \frac{3z_1}{-2} \qquad \text{(comparing coefficients)},$$

$$\therefore\ x_1 = 0, \quad y_1 = -\tfrac{3}{4}z_1.$$

Since (x_1, y_1, z_1) lies on the ellipsoid (i), these values can be substituted in equation (i), giving

$$\tfrac{9}{8}z_1^2 + 3z_1^2 = 1, \ \text{ i.e. } \tfrac{33}{8}z_1^2 = 1.$$

$$\therefore\ z_1 = -2\sqrt{\tfrac{2}{33}} \qquad \text{(using negative root)}.$$

$$\therefore\ y_1 = \tfrac{3}{2}\sqrt{\tfrac{2}{33}}.$$

Hence one of the required tangent planes is

$$3\sqrt{\tfrac{2}{33}}\,.\,y - 6\sqrt{\tfrac{2}{33}}\,.\,z = 1 \qquad \text{[using (ii)]},$$

i.e.
$$y - 2z = \tfrac{1}{3}\sqrt{\tfrac{33}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(iii)}$$

The equation of the plane half-way between the plane (iii) and the plane

$$y - 2z = 0$$

is
$$y - 2z = \tfrac{1}{6}\sqrt{\tfrac{33}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(iv)}$$

If (α, β, γ) be the centre of the conic that the plane (iv) forms with the ellipsoid, then by Part (i) of the question, the plane (iv) will be equivalent to

$$\alpha x + 2\beta y + 3\gamma z = \alpha^2 + 2\beta^2 + 3\gamma^2. \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(v)}$$

Comparing (iv) and (v),

$$\frac{\alpha}{0} = \frac{2\beta}{1} = \frac{3\gamma}{-2} = \frac{\alpha^2 + 2\beta^2 + 3\gamma^2}{\frac{1}{6}\sqrt{\frac{33}{2}}}.$$

$$\therefore\ \alpha = 0 \ \text{ and } \ 2\beta = -\tfrac{3}{2}\gamma = (2\beta^2 + 3\gamma^2)/\tfrac{1}{6}\sqrt{\tfrac{33}{2}}.$$

Hence
$$\beta = -\tfrac{3}{4}\gamma \ \text{ and } \ -\tfrac{3}{2}\gamma = \frac{\frac{9}{8}\gamma^2 + 3\gamma^2}{\frac{1}{6}\sqrt{\frac{33}{2}}} = \frac{\frac{33}{8}\gamma^2}{\frac{1}{6}\sqrt{\frac{33}{2}}}.$$

$$\therefore\ \tfrac{33}{8}\gamma = -\tfrac{3}{2} \times \tfrac{1}{6}\sqrt{\tfrac{33}{2}}. \quad \therefore\ \gamma = -\sqrt{\tfrac{2}{33}}, \ \text{and} \ \beta = \tfrac{3}{4}\sqrt{\tfrac{2}{33}}.$$

Hence the required point is

$$(0, \ \tfrac{3}{4}\sqrt{\tfrac{2}{33}}, \ -\sqrt{\tfrac{2}{33}}).$$

Example 7 (L.U.).—Define the polar plane of a point with respect to an ellipsoid, and find the equation of the polar plane of (x_1, y_1, z_1) with respect to the ellipsoid $ax^2 + by^2 + cz^2 = 1$.

Show that the polar planes of all points on the line $\dfrac{x-2}{1} = \dfrac{y-1}{2} = \dfrac{z+1}{3}$, with respect to the ellipsoid $x^2 + 2y^2 + 3z^2 = 6$, pass through a fixed line, and find the equations of the fixed line in standard form.

The polar plane of the point P, with respect to a given ellipsoid, is the plane that contains all the points of contact, real or imaginary, of the tangents drawn from P to the ellipsoid.

Let (α, β, γ) be a point on the ellipsoid

$$ax^2 + by^2 + cz^2 = 1. \quad \ldots \quad \text{(i)}$$

Then the tangent plane at this point to the ellipsoid (i) is

$$ax\alpha + by\beta + cz\gamma = 1. \quad \ldots \quad \text{(ii)}$$

But if this plane pass through the point (x_1, y_1, z_1), then

$$ax_1\alpha + by_1\beta + cz_1\gamma = 1. \quad \ldots \quad \text{(iii)}$$

But equation (iii) is the condition that the plane

$$ax_1x + by_1y + cz_1z = 1$$

shall pass through (α, β, γ), and hence the required equation is

$$axx_1 + byy_1 + czz_1 = 1.$$

Let (x_1, y_1, z_1) be any point of the line

$$\frac{x-2}{1} = \frac{y-1}{2} = \frac{z+1}{3} = r \quad \text{(say).} \quad \ldots \quad \text{(iv)}$$

Then
$$\frac{x_1-2}{1} = \frac{y_1-1}{2} = \frac{z_1+1}{3} = r,$$

i.e.
$$x_1 = 2 + r,\ y_1 = 1 + 2r,\ z_1 = -1 + 3r. \quad \ldots \quad \text{(v)}$$

The polar plane of (x_1, y_1, z_1), with respect to the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 6,$$

is
$$xx_1 + 2yy_1 + 3zz_1 - 6 = 0. \quad \ldots \quad \text{(vi)}$$

Using the values (v) in (vi), since (x_1, y_1, z_1) lies on the line (iv),

$$(2 + r)x + 2(1 + 2r)y + 3(3r - 1)z - 6 = 0,$$

i.e.
$$2x + 2y - 3z - 6 + r(x + 4y + 9z) = 0. \quad \ldots \quad \text{(vii)}$$

Now if $P_1 = 0$ and $P_2 = 0$ represent two planes, and λ be any constant, then $P_1 + \lambda P_2 = 0$ represents any plane through their line of intersection. Therefore the equation (vii) represents a plane through the line given by

$$\left.\begin{aligned} 2x + 2y - 3z - 6 &= 0 \\ \text{and} \quad x + 4y + 9z &= 0 \end{aligned}\right\}. \quad \ldots \quad \text{(viii)}$$

Hence all polar planes pass through the line given by the equations (viii).

Let l, m, n be the direction cosines of the line represented by equations (viii).

Since the line lies in the planes given in (viii),

$$2l + 2m - 3n = 0,$$

$$l + 4m + 9n = 0.$$

$$\therefore \frac{l}{30} = \frac{-m}{21} = \frac{n}{6}, \quad \text{i.e. } \frac{l}{10} = \frac{m}{-7} = \frac{n}{2}.$$

By inspection of equations (viii) the point $(4, -1, 0)$ is a point on the required line. Hence its standard equation is

$$\frac{x-4}{10}=\frac{y+1}{-7}=\frac{z}{2}.$$

Example 8 (L.U.).—Chords of the ellipsoid $3x^2+6y^2+2z^2=6a^2$ are drawn equally inclined to the axes. Show that their mid-points lie on a plane.

Find the equation of the projection on the xy-plane of the curve of intersection of the above plane with the ellipsoid.

A chord equally inclined to the axes will have as its equation

$$\frac{x-x_1}{1}=\frac{y-y_1}{1}=\frac{z-z_1}{1}=r \qquad \text{(say)}.$$

$$\therefore\ x=x_1+r,\ y=y_1+r,\ z=z_1+r. \qquad \text{(i)}$$

Where this chord intersects the ellipsoid

$$3x^2+6y^2+2z^2-6a^2=0, \qquad \text{(ii)}$$

by substitution from (i) in (ii),

$$3(x_1+r)^2+6(y_1+r)^2+2(z_1+r)^2-6a^2=0,$$

i.e. $$11r^2+2r(3x_1+6y_1+2z_1)+3x_1^2+6y_1^2+2z_1^2-6a^2=0. \qquad \text{(iii)}$$

If (x_1, y_1, z_1) be the mid-point of the chord, then the roots of the equation (iii) as a quadratic in r must be equal and opposite,

i.e. $$3x_1+6y_1+2z_1=0.$$

Therefore the mid-points of all such chords must lie on the plane

$$3x+6y+2z=0. \qquad \text{(iv)}$$

It can be seen from a diagram that the equation of the projected curve on the plane $z=0$ can be obtained by eliminating z between the equations (ii) and (iv).

Using $2z=-(3x+6y)$ in equation (ii), the required equation is

$$3x^2+6y^2+\tfrac{1}{2}(3x+6y)^2-6a^2=0,\ z=0;$$

i.e. $$6x^2+12y^2+9x^2+36xy+36y^2-12a^2=0,\ z=0;$$

i.e. $$15x^2+36xy+48y^2=12a^2,\ z=0;$$

i.e. $$5x^2+12xy+16y^2=4a^2,\ z=0.$$

The Prolate Spheroid and Oblate Spheroid

18. If in the case of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, two of the quantities, say b and c, be equal, then all sections parallel to $x=0$ will be circles, and therefore the surface is that formed by the revolution of the ellipse $x^2/a^2+y^2/b^2=1$ about the axis of x.

The surface formed by the revolution of an ellipse about its major axis is known as a *prolate spheroid*, and that formed by its revolution about the minor axis is known as an *oblate spheroid.*

When $a = b = c$ in the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, the equation becomes $x^2 + y^2 + z^2 = a^2$, i.e. the equation of a sphere with its centre at the origin O, and radius a.

Hyperboloids and Paraboloids

19. The surface whose standard equation is

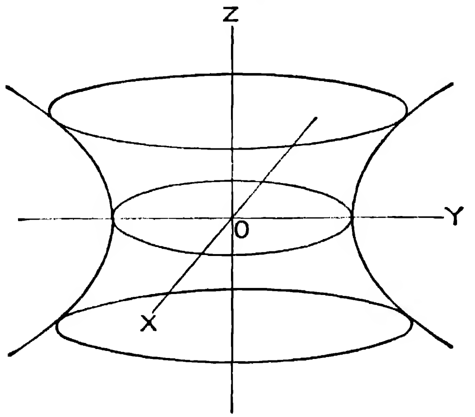
$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$$

is called a *hyperboloid of one sheet.*

From the equation it can be seen that the intercepts on OX and OY are real, and the intercept on OZ is imaginary, whilst the surface is symmetrical about the three co-ordinate planes.

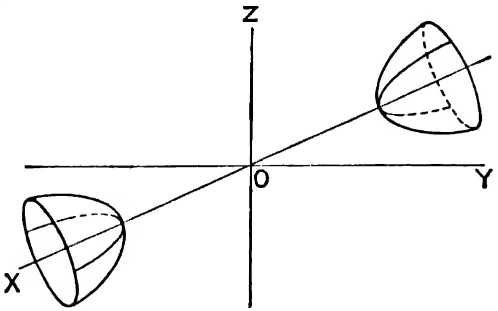
The sections by planes parallel to $x = 0$ and $y = 0$ are hyperbolas, and the sections parallel to the plane $z = 0$ are ellipses whose axes gradually increase as the plane is moved farther from the plane $z = 0$.

From these results the surface is as shown in the diagram.



The surface whose equation is $x^2/a^2 - y^2/b^2 - z^2/c^2 = 1$ is known as a *hyperboloid of two sheets*, and from the equation it can be seen that it has real intercepts on OX and imaginary ones on OY and OZ. The surface is symmetrical about the co-ordinate planes. Also, sections by planes parallel to $y = 0$ and $z = 0$ are hyperbolas, and sections parallel to $x = 0$ cut the surface in ellipses, providing they are of the

form $x = k$, where $k^2 > a^2$. From these results the surface is as shown in the diagram.



The surface represented by the equation $x^2/a + y^2/b = 2z$ is known as a *paraboloid*, and if a and b are of the same sign it is an *elliptic paraboloid*, whilst if a and b are of opposite signs it is a *hyperbolic paraboloid*. In the case of the elliptic paraboloid, all sections parallel to $x = 0$ and $y = 0$ are parabolas, and all sections parallel to $z = 0$ are ellipses; whilst in the case of the hyperbolic paraboloid, all sections parallel to $x = 0$ and $y = 0$ are parabolas, and all sections parallel to $z = 0$ are hyperbolas.

Example 9 (L.U.).—Obtain the equation to the tangent plane, and the equation to the normal to the surface $az = xy$ at the point $(a\alpha, a\beta, a\alpha\beta)$.

Show that the tangent plane intersects the surface in two straight lines, and give their equations.

Working from first principles, any line through $(a\alpha, a\beta, a\alpha\beta)$ is given by

$$\frac{x - a\alpha}{l} = \frac{y - a\beta}{m} = \frac{z - a\alpha\beta}{n} = r \qquad \text{(say)}, \quad . \quad . \quad \text{(i)}$$

$$\therefore\ x = a\alpha + rl,\ y = a\beta + rm,\ z = a\alpha\beta + rn.$$

Using these values in the equation to the surface

$$az = xy, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

the points of intersection of the line (i) with the surface (ii) are given by

$$a(a\alpha\beta + rn) = (a\alpha + rl)(a\beta + rm),$$

i.e. $$a^2\alpha\beta + arn = a^2\alpha\beta + ar\alpha m + ar\beta l + r^2lm,$$

i.e. $$r^2lm + r(a\beta l + a\alpha m - an) = 0. \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

One value of r in (iii) is zero, and if the line (i) is to be a tangent line to the surface (ii), then a second value of r in equation (iii) must be zero,

i.e. $$a\beta l + a\alpha m - an = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Eliminating l, m, n between equations (i) and (iv), all the tangent lines at $(a\alpha, a\beta, a\alpha\beta)$ must lie on the surface

$$a\beta(x - a\alpha) + a\alpha(y - a\beta) - a(z - a\alpha\beta) = 0,$$

i.e.
$$a\beta x + a\alpha y - az = a^2\alpha\beta,$$

which is a plane, being linear in x, y, and z. Hence the equation of the tangent plane at $(a\alpha, a\beta, a\alpha\beta)$ is

$$\beta x + \alpha y - z = a\alpha\beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

From this result, the normal will have direction cosines proportional to $(\beta, \alpha, -1)$, therefore the equation of the normal at $(a\alpha, a\beta, a\alpha\beta)$ is

$$\frac{x - a\alpha}{\beta} = \frac{y - a\beta}{\alpha} = \frac{z - a\alpha\beta}{-1}.$$

Where the tangent plane (v) cuts the surface (ii), by substituting for z from (v) in (ii),

$$a(\beta x + \alpha y - a\alpha\beta) = xy,$$

i.e.
$$xy - a\beta x - a\alpha y + a^2\alpha\beta = 0,$$

i.e.
$$(x - a\alpha)(y - a\beta) = 0,$$

i.e.
$$x = a\alpha \quad \text{and} \quad y = a\beta.$$

Hence the tangent plane cuts the given surface in two lines given by

$$x = a\alpha, \quad \beta x + \alpha y - z = a\alpha\beta,$$

and
$$y = a\beta, \quad \beta x + \alpha y - z = a\alpha\beta,$$

which can be written in the standard form as

$$\frac{x - a\alpha}{0} = \frac{y}{1} = \frac{z}{\alpha},$$

$$\frac{x}{1} = \frac{y - a\beta}{0} = \frac{z}{\beta}.$$

Example 10 (L.U.).—Show that two straight lines through any point of the surface $xy + z^2 = c^2$ can be drawn lying wholly on the surface.

Show also that the points where these straight lines are perpendicular are the intersections of the sphere $x^2 + y^2 + z^2 = c^2$, with this surface.

Let $P \equiv (x_1, y_1, z_1)$ be any point on the surface

$$xy + z^2 = c^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Then
$$x_1y_1 + z_1^2 = c^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Any line through P is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \quad \text{(say)}, \quad . \quad . \quad . \quad \text{(iii)}$$

i.e.
$$x = x_1 + rl, \; y = y_1 + rm, \; z = z_1 + rn. \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Where the line (iii) cuts the surface (i),

$$(x_1 + rl)(y_1 + rm) + (z_1 + rn)^2 = c^2,$$

i.e.
$$x_1y_1 + r(mx_1 + ly_1) + r^2lm + z_1^2 + 2rnz_1 + r^2n^2 = c^2,$$

i.e. using (ii),
$$r^2(lm + n^2) + r(mx_1 + ly_1 + 2nz_1) = 0. \quad . \quad . \quad . \quad . \quad \text{(v)}$$

If the line (iii) lie entirely on the surface (i), the result (v) must be true for all values of r, since all points on the line must lie on the surface.

$$\therefore\ lm + n^2 = 0, \qquad \text{(vi)}$$

and
$$mx_1 + ly_1 + 2nz_1 = 0. \qquad \text{(vii)}$$

From (vii)
$$n^2 = \frac{1}{4z_1^2}(mx_1 + ly_1)^2,$$

and using this in (vi), $\quad 4lmz_1^2 + (mx_1 + ly_1)^2 = 0,$

i.e.
$$l^2y_1^2 + lm(4z_1^2 + 2x_1y_1) + m^2x_1^2 = 0. \qquad \text{(viii)}$$

The equation (viii) is a quadratic in l/m giving two real values of l/m, and hence two corresponding values can be found for n/m. Thus, there are two lines through any point on the surface $xy + z^2 = c^2$ lying wholly on the surface.

If (l_1, m_1, n_1), (l_2, m_2, n_2) be the proportional direction cosines of the two lines, then, from (viii),

$$\frac{l_1l_2}{m_1m_2} = \frac{x_1^2}{y_1^2}, \quad \frac{l_1}{m_1} + \frac{l_2}{m_2} = -\frac{(4z_1^2 + 2x_1y_1)}{y_1^2}. \qquad \text{(ix)}$$

Also, from (vii),

$$n_1 = -\frac{1}{2z_1}(m_1x_1 + l_1y_1), \quad n_2 = -\frac{1}{2z_1}(m_2x_1 + l_2y_1).$$

$$\therefore\ n_1n_2 = \frac{1}{4z_1^2}\{m_1m_2x_1^2 + (m_1l_2 + l_1m_2)x_1y_1 + l_1l_2y_1^2\}.$$

$$\therefore\ \frac{n_1n_2}{m_1m_2} = \frac{1}{4z_1^2}\left\{x_1^2 + \left(\frac{l_2}{m_2} + \frac{l_1}{m_1}\right)x_1y_1 + \frac{l_1l_2}{m_1m_2}y_1^2\right\}$$

$$\frac{1}{4z_1^2}\{x_1^2 - (4z_1^2 + 2x_1y_1)\,.\,\frac{x_1}{y_1} + x_1^2\} \qquad \text{[using (ix)]}$$

$$= \frac{1}{4z_1^2}\left\{-4z_1^2\,.\,\frac{x_1}{y_1}\right\} = -\frac{x_1}{y_1}. \qquad \text{(x)}$$

Hence, from (ix) and (x),

$$\frac{l_1l_2}{m_1m_2} + 1 + \frac{n_1n_2}{m_1m_2} = \frac{x_1^2}{y_1^2} + 1 - \frac{x_1}{y_1},$$

i.e.
$$l_1l_2 + m_1m_2 + n_1n_2 = \frac{m_1m_2}{y_1^2}\{x_1^2 + y_1^2 - x_1y_1\}$$

$$= \frac{m_1m_2}{y_1^2}\{x_1^2 + y_1^2 + z_1^2 - c^2\} \text{ [using (ii)].}$$

But the two lines are perpendicular if

$$l_1l_2 + m_1m_2 + n_1n_2 = 0,$$

i.e. if
$$x_1^2 + y_1^2 + z_1^2 - c^2 = 0,$$

i.e. if (x_1, y_1, z_1) lie on the surface $x^2 + y^2 + z^2 = c^2$, which is the equation of a sphere.

Thus, the points where the lines are perpendicular are the intersection of the sphere $x^2 + y^2 + z^2 = c^2$ and the surface.

EXAMPLES ON CHAPTER XIII

All the following questions are taken from London University examination papers.

1. Planes are drawn through the point A (2, 3, 2) inclined at 30° to the diameter through A of the sphere

$$x^2 + y^2 + z^2 - 2x - 2y - 2z - 21 = 0.$$

Show that the locus of centres of sections of the sphere by these planes is the circle given by

$$(x - 1)(x - 2) + (y - 1)(y - 3) + (z - 1)(z - 2) = 0, \ 2x + 4y + 2z = 11.$$

2. Obtain the equation of the sphere on the straight line joining the points (x_1, y_1, z_1), (x_2, y_2, z_2) as diameter.

A sphere of radius R passes through the origin. Show that the extremities of the diameter parallel to the x-axis lie one on each of the spheres

$$x^2 + y^2 + z^2 \pm 2Rx = 0.$$

3. Find the condition that the plane $lx + my + nz = p$ may touch the sphere $(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2$.

Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point $(1, -2, 1)$ and also cuts orthogonally the sphere

$$x^2 + y^2 + z^2 - 4x + 6y + 4 = 0.$$

4. A system of spheres is such that any two spheres of the system touch at a fixed point O. Show that any sphere which cuts every sphere of the system orthogonally passes through O. Show also that, if this sphere pass through a certain fixed point A, it passes through a certain fixed circle.

5. Find the co-ordinates of the centre of the circle circumscribing the triangle whose vertices are (2, 0, 4), (2, 4, 2), (0, 2, 4).

Find the equation of the sphere which passes through these three points and also through the origin.

6. Show that, by a suitable choice of axes, the equations of any two non-intersecting straight lines can be written in the form

$$y = mx, \ z = c; \quad y = -mx, \ z = -c.$$

The shortest distance between two skew lines is AB. P is any point on the line through A, and M is the mid-point of AP; Q is any point on the line through B, and N is the mid-point of BQ. Prove that, if OM, ON be perpendicular, where O is the mid-point of AB, then AQ and BP are perpendicular. Show also that the radical plane of the spheres with diameters MN and PQ is the polar plane of O with respect to the second sphere.

7. (i) Find the co-ordinates of the points in which the straight line

$$\frac{x - 3}{1} = \frac{y - 5}{2} = \frac{z - 5}{-1}$$

cuts the sphere $x^2 + y^2 + z^2 = 49$.

(ii) Find also the distance from the point (3, 5, 5) to the point in which the given line meets the plane through the point (4, 1, -8) to which it is a normal.

8. Show that the equations of any two lines can be put in the form

$$y = x\tan\alpha,\ z = a;\quad y = -x\tan\alpha,\ z = -a.$$

Two lines are intersected by their shortest distance at the points A and A_1, and the mid-point of AA_1 is O. P is a point on the line through A, and P_1 is a point on the line through A_1. If $AP \,.\, A_1P_1$ has either of the values $a^2\sec^2\alpha$ or $-a^2\operatorname{cosec}^2\alpha$, show that PP_1 touches the sphere whose centre is O and radius a.

9. Prove that the polar plane of the point (x_1, y_1, z_1), with respect to the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,$$

is $$x(x_1 + u) + y(y_1 + v) + z(z_1 + w) + ux_1 + vy_1 + wz_1 + d = 0.$$

Find the polar plane of the point P (1, 2, 3) with respect to the sphere whose centre is C (2, -2, 1) and whose radius is 3.

Find also the foot Q of the perpendicular drawn from C to this plane, and verify that $CP \,.\, CQ = 9$.

10. (i) Show that the sphere

$$(x - \alpha - p)(x - \alpha) + (y - \beta - q)(y - \beta) + (z - \gamma - r)(z - \gamma) = a^2$$

intersects the sphere $(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = a^2$ along a great circle of the second sphere.

(ii) Find the equation of the sphere of minimum radius which belongs to the coaxal system defined by the spheres

$$x^2 + y^2 + z^2 + 2x - 2y + 4z + 2 = 0,$$
$$x^2 + y^2 + z^2 + 4x + 2y - 4z = 0.$$

[*N.B.*—A *coaxal system* of spheres is such that any pair has the same radical plane.]

11. Show that the equation of a sphere can be written in the form

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

Find the equation of the sphere which passes through the circle

$$x^2 + y^2 + 2x = 0,\ z = 0,$$

and the point (1, 2, 1), and verify that it cuts the plane $y = 4$ in a circle of unit radius.

12. Find the equation of the tangent plane to the sphere

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2$$

at the point (x_1, y_1, z_1).

Find the equation of a sphere which touches the sphere

$$x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$$

at the point (1, 2, -2) and passes through the origin.

13. The lengths of the edges of a rectangular parallelepiped are a, b, c. A point moves so that the sum of the squares of its distances from the six faces of the parallelepiped is constant and equal to m^2d^2, where d is the length of the diagonal of the parallelepiped. Show that the locus of the point is a sphere, and calculate the volume of this sphere. Show also that the sphere is real if, and only if, m is not less than $1/\sqrt{2}$.

14. Determine the equation of the sphere through the points (0, 1, 3), (1, 2, 4), (2, 3, 1), and (3, 0, 2).

Find also the equation of the shadow of this sphere thrown on the plane $z = 0$ by a point source of light at the point (0, 0, 6).

15. Show that the equation

$$x^2 + y^2 + z^2 + 2ax + 2by + 2cz + d + 2\lambda(\alpha x + \beta y + \gamma z + \delta) = 0$$

represents a sphere through the circle common to the sphere

$$x^2 + y^2 + z^2 + 2ax + 2by + 2cz + d = 0$$

and the plane $\alpha x + \beta y + \gamma z + \delta = 0$.

Find the equation of the two spheres through the common circle of the sphere $x^2 + y^2 + z^2 + 2x + 2y = 0$ and the plane $x + y + z + 4 = 0$, and which intersect the plane $x + y = 0$ in circles of radius 3 units.

16. Obtain the equation of the tangent plane to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at the point (x_1, y_1, z_1).

If the normal at any point P of the ellipsoid meets the plane $z = 0$ at G, and if GP be produced to Q so that PQ = GP, show that the locus of Q is

$$\frac{a^2x^2}{(a^2 + c^2)^2} + \frac{b^2y^2}{(b^2 + c^2)^2} + \frac{z^2}{4c^2} = 1.$$

17. Find the condition that the plane $lx + my + nz = p$ shall touch the ellipsoid $ax^2 + by^2 + cz^2 = 1$.

Find the equations of the two planes which can be drawn through the line $3x + 4y + 9z = 20$, $x = 3z$, to touch the ellipsoid $x^2 + 4y^2 + 9z^2 = 10$. Find also the equations of the chord of contact of these planes.

18. Write down the equation of the tangent plane at $P \equiv (x_1, y_1, z_1)$ to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and show that, if the normal at P meets the plane $x = 0$ at G, then PG is of length a^2/p, where p is the distance of the tangent plane from the centre of the ellipsoid.

If through G a line GH be drawn parallel to the x-axis so that PG = GH, show that the locus of H is the conicoid

$$\frac{x^2}{a^2} - \frac{y^2}{a^2 - b^2} - \frac{z^2}{a^2 - c^2} = 1.$$

19. Obtain the equations of the normal at any point of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

A normal is drawn to this ellipsoid at any point P on the section by the plane $z=k$. If Q be the point in which this normal meets the plane $x=0$, show that the locus of Q is a line parallel to the y-axis, and the distance between the extreme positions of Q is

$$\frac{2}{bc}(a^2-b^2)\sqrt{(c^2-k^2)}.$$

20. Find the equation of the polar plane of the point (α, β, γ) with respect to the ellipsoid $ax^2+by^2+cz^2=1$.

Show that the poles of all planes through the line

$$\frac{x-2}{1}=\frac{y-1}{2}=\frac{z+3}{3},$$

with respect to the ellipsoid $x^2+2y^2+3z^2=6$, lie on a fixed line, and find the equations of the fixed line in standard form.

21. Find the equation of the polar of the point (x_1, y_1, z_1) with respect to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Show that the intersection of the plane $z=0$, and the polar plane with respect to this ellipsoid of any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=c,$$

is a tangent to the ellipse $\quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1,\ z=0.$

22. Find the equations of the tangent plane and normal to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

at the point (x_1, y_1, z_1).

The normal at the point P to the ellipsoid $x^2+4y^2+z^2/4=1$ meets it again at the point Q. The foot of the perpendicular from O, the centre of the ellipsoid, to the tangent plane at P, is M. If PQ . OM$^3=-2$, show that P also lies on the ellipsoid $x^2+64y^2+z^2/64=1$.

23. Find the equation of the tangent plane and the normal to the ellipsoid $ax^2+by^2+cz^2=1$ at the point (x_1, y_1, z_1).

The point P on the ellipsoid $3x^2+2y^2+4z^2=a^2$ is such that the normal at P intersects the plane $z=0$ at a point lying on the parabola $y^2=4hx$. Show that the tangent plane at P intersects the plane $z=0$ in a line which touches the parabola $4hy^2+3a^2x=0$.

24. Find the equation of the tangent plane and the normal at the point (x_1, y_1, z_1) on the ellipsoid $ax^2+by^2+cz^2=1$.

At the points where the plane $x + y + z = 0$ intersects the ellipsoid $x^2 + 2y^2 + 3z^2 = 6$, normals to the ellipsoid are drawn. Show that these normals meet the plane $z = 0$ on the ellipse $3x^2 + 9xy + 15y^2 = 2$.

25. Find the equations to the tangent plane and normal to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at the point (X, Y, Z) upon it.

If the axes cut the tangent plane in L, M, N, show that the area of the triangle LMN is $\dfrac{a^2b^2c^2}{2pXYZ}$, where p is the length of the perpendicular from O on to the tangent plane.

26. Obtain the condition that the plane $lx + my + nz = 1$ shall touch the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Prove that the plane $x + y + nz = 1$ will intersect the ellipsoid

$$3x^2 + 2y^2 + z^2 = 1$$

in real points only if $6n^2 > 1$.

27. The normal at P (x_1, y_1, z_1) to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

meets the plane $z = 0$ at Q and meets the surface again at R. Prove that, if PR = 4PQ, the locus of P is on the cone

$$\frac{x^2(2c^2 - a^2)}{a^6} + \frac{y^2(2c^2 - b^2)}{b^6} + \frac{z^2}{c^4} = 0.$$

28. Write down the equations of the tangent plane and normal at the point (α, β, γ), on the conicoid $f(x, y, z) = 0$.

Show that the tangent planes to the surface $xy + yz + zx = r^2$, at points on the intersection of the surface with the plane $x + y + z = r\sqrt{3}$, are also tangent planes to the sphere $x^2 + y^2 + z^2 = r^2$.

29. Show that the square of the shortest distance from the point P (α, β, γ) to the straight line whose equations are

$$\frac{x - a}{l} = \frac{y - b}{m} = \frac{z - c}{n}$$

(l, m, n being the actual direction cosines of the line)

is $[m(\gamma - c) - n(\beta - b)]^2 + [n(\alpha - a) - l(\gamma - c)]^2 + [l(\beta - b) - m(\alpha - a)]^2$.

Find the equation of the circular cylinder of radius 2, whose axis is the line

$$\frac{x - 1}{3} = \frac{y - 3}{2} = \frac{z}{2}.$$

30. Show that the equation of the tangent plane at the point (α, β, γ) on the surface $xy = cz$ is $\beta x + \alpha y - cz = c\gamma$, and find the equations of the normal at this point.

Show that this tangent plane touches the sphere of radius r with centre at the origin if $r^2(\alpha^2 + \beta^2 + c^2) = c^2\gamma^2$, and that the normal at (α, β, γ) to the given surface is a tangent line to the sphere of radius R, with centre at the origin. if $R^2(\alpha^2 + \beta^2 + c^2) = (\alpha^2 + \beta^2)(\alpha^2 + \beta^2 + \gamma^2 + c^2)$.

31. Find the perpendicular distance of the point (f, g, h) from the line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}.$$

Find the equation of the cone whose whole vertical angle is 90°, which has its vertex at the origin, and its axis along the line $x = -2y = z$, and show that the plane $z = 0$ cuts the cone in two straight lines inclined at angle of $\cos^{-1} \frac{4}{5}$.

32. Show that the surface whose equation is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = f(lx + my + nz)$$

is a surface of revolution, and find the equations of the axis of revolution. (f means 'function'.)

Show that the surface whose equation is

$$(x + y + z)^2 = x^2 + y^2 + z^2$$

is a right circular cone, and find the vertical angle of the cone.

33. Find the equation of the tangent plane at the point $(a\alpha, a\beta, a\gamma)$ on the surface $2xy = z^2 - a^2$.

Show that this plane intersects the surface in two straight lines, and find their equations.

CHAPTER XIV

Area under a Curve, Volume of Revolution, etc.

Area under a Curve

1. The area under the curve $y = f(x)$ between the ordinates $x = a$ and $x = b$ is given by

$$\int_a^b y\,dx = \int_a^b f(x)\,dx.$$

By splitting up the area into a very large number of small areas by means of equidistant ordinates δx apart, it can be seen that the area under the curve is also equal to $\underset{\delta x \to 0}{\mathrm{Lt}} \sum_{x=a}^{b} y\,\delta x$. Hence this latter expression is equal to $\int_a^b y\,dx$, and vice versa.

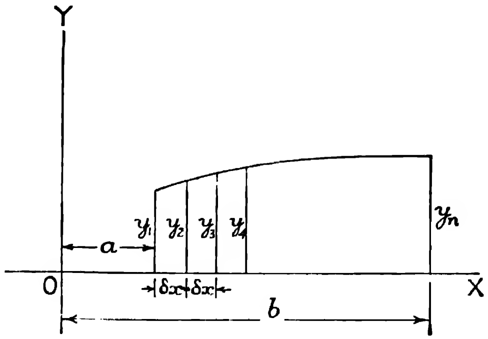
In the case where the curve is entirely above the x-axis, and $b > a$, all the ordinates are positive and δx is positive, and thus the area obtained will be positive; it will be a negative area in this case if $a > b$, as δx will then be negative.

For a portion of curve entirely below the x-axis and $b > a$, all the ordinates will be negative and δx positive, and the area under the curve will be negative; it will be a positive area in this case if $a > b$, as δx will then be negative. Hence, in taking $\int_a^b y\,dx$, with $b > a$, the result obtained for the area under the curve will be the sum of the portions above the x-axis diminished by the sum of the portions of area below the x-axis.

If the actual total area be required, it will be necessary to find where the curve cuts the x-axis, and find the area from $x = a$ to the first point of intersection of the curve and the x-axis, then the area from this point of intersection to the next point of intersection, and so on. The actual total area will then be given by the numerical sum of these areas.

2. Theorem.—To find the mean value of y between $x = a$ and $x = b$, where $y = f(x)$.

The area under the curve $y = f(x)$ is split up into small areas by means of the ordinates $y_1, y_2, y_3, \ldots, y_n$, at a small distance δx apart, where y_1 is the ordinate at $x = a$, etc.



The mean value of y is given by

$$\underset{n \to \infty}{\mathrm{Lt}} \frac{y_1 + y_2 + y_3 + \ldots + y_n}{n} = \underset{n \to \infty}{\mathrm{Lt}} \frac{\delta x(y_1 + y_2 + \ldots + y_n)}{n\,\delta x}$$

$$= \underset{\delta x \to 0}{\mathrm{Lt}} \left\{ \frac{\sum_{x=a}^{x=b} y\,\delta x}{b - a + \delta x} \right\} = \frac{\underset{\delta x \to 0}{\mathrm{Lt}} \sum_{x=a}^{x=b} y\,\delta x}{\underset{\delta x \to 0}{\mathrm{Lt}} (b - a + \delta x)}$$

$$= \frac{1}{b-a}\int_a^b y\,dx = \frac{\text{area under the curve}}{\text{base}}.$$

3. Theorem.—To find the area included between the curve $r = f(\theta)$ and the two radius vectors given by $\theta = \theta_1$ and $\theta = \theta_2$.

O is the pole of co-ordinates, and OX the initial line. APQB is the curve such that

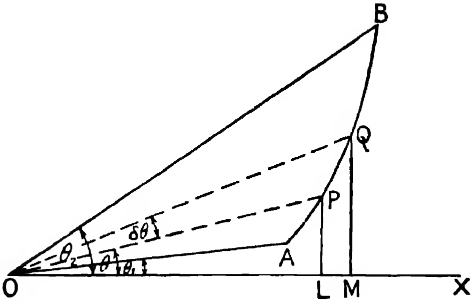
$$\angle AOX = \theta_1,\ \angle BOX = \theta_2,\ P \equiv (r, \theta), \text{ and } Q \equiv (r + \delta r,\ \theta + \delta\theta).$$

The element of area POQ $= \frac{1}{2}r^2\,\delta\theta$.

$$\therefore \text{ Total area} = \underset{\delta\theta \to 0}{\mathrm{Lt}} \sum_{\theta=\theta_1}^{\theta_2} \tfrac{1}{2}r^2\,\delta\theta = \int_{\theta_1}^{\theta_2} \tfrac{1}{2}r^2\,d\theta.$$

A useful result can be obtained in the case of a curve given in the parametric form as follows:

In the diagram let P and Q have Cartesian co-ordinates (x, y) and $(x + \delta x, y + \delta y)$ respectively. PL, QM are the ordinates of P and Q.



Since arc PQ can be taken to be a straight line in the limit when $\delta\theta \to 0$, the element of area POQ

$$= \text{area } \triangle OQM - \text{area } \triangle POL - \text{area trapezium PLMQ}$$
$$= \tfrac{1}{2}(x + \delta x)(y + \delta y) - \tfrac{1}{2}xy - \tfrac{1}{2}\{y + (y + \delta y)\}\,\delta x$$
$$= \tfrac{1}{2}(x\,\delta y - y\,\delta x).$$

$$\therefore \textbf{ Area sector OAB} = \operatorname*{Lt}_{\substack{\delta x \to 0 \\ \delta y \to 0}} \sum_{\theta = \theta_1}^{\theta = \theta_2} \tfrac{1}{2}(x\,\delta y - y\,\delta x)$$
$$\tfrac{1}{2}\int_{\theta = \theta_1}^{\theta = \theta_2} (x\,dy - y\,dx).$$

Example 1.—Find the area of the ellipse given by the equations

$$x = 5\cos\phi, \quad y = 3\sin\phi.$$

The ellipse is completely traced out once if ϕ be taken from 0 to 2π, and these must be the limits for ϕ in finding the area.

From $x = 5\cos\phi$, $\qquad dx = -5\sin\phi\,d\phi$,

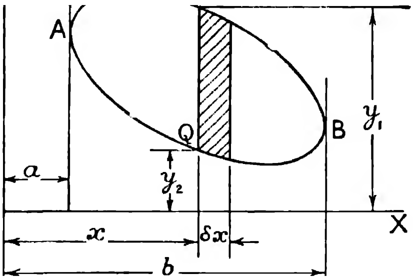
and from $y = 3\sin\phi$, $\qquad dy = 3\cos\phi\,d\phi$.

Hence the required area $= \tfrac{1}{2}\displaystyle\int_{\phi=0}^{\phi=2\pi}(x\,dy - y\,dx)$

$$= \tfrac{1}{2}\int_0^{2\pi}\{5\cos\phi\,.\,3\cos\phi\,d\phi - 3\sin\phi(-5\sin\phi)\,d\phi\}$$
$$= \frac{15}{2}\int_0^{2\pi}(\cos^2\phi + \sin^2\phi)\,d\phi = \frac{15}{2}\int_0^{2\pi}d\phi$$
$$= \frac{15}{2}\Big[\phi\Big]_0^{2\pi} = \frac{15}{2} \times 2\pi = 15\pi \text{ square units.}$$

4. Theorem.—To find the area of a closed curve which is given in Cartesian co-ordinates.

The closed curve is shown in the diagram with A and B points on it such that the tangents at A and B are parallel to OY. At A and B the values of x are a and b respectively. P is any point on the curve, and the ordinate through P meets the curve again at Q, the ordinates of P and Q being respectively y_1 and y_2, and their abscissa being x.



A line is taken parallel to PQ and at a distance δx from it to form the shaded element of area. This shaded element of area $= (y_1 - y_2)\,\delta x$, when $\delta x \to 0$. Therefore the area of the closed curve = the sum of all such elements between $x = a$ and $x = b$

$$= \underset{\delta x \to 0}{\text{Lt}} \sum_{x=a}^{x=b} (y_1 - y_2)\,\delta x = \int_a^b (y_1 - y_2)\,dx.$$

N.B.—The value of $(y_1 - y_2)$ will be found from the equation of the curve, which, if a single curve, must be a quadratic in y, and the values of a and b will be given when $y_1 - y_2 = 0$.

Example 2.—Find the area of either half of the ellipse $13x^2 + 10xy + 13y^2 = 72$ as divided by the y-axis.

The equation of the curve can be written as

$$13y^2 + 10xy + (13x^2 - 72) = 0.$$

Solving this as a quadratic in y for its roots y_1 and y_2,

$$y = \{-5x \pm \sqrt{(25x^2 - 169x^2 + 936)}\}/13$$
$$= \{-5x \pm \sqrt{(936 - 144x^2)}\}/13.$$

$$\therefore\ y_1 - y_2 = 2\sqrt{(936 - 144x^2)}/13.$$

Also $\quad y_1 - y_2 = 0$, when $936 - 144x^2 = 0$,

i.e. when $\quad x = \pm\sqrt{(13/2)}$.

Since only half the ellipse is required, the area will be taken from $x = 0$ to $x = \sqrt{(13/2)}$.

Thus the area of the half-ellipse is $\dfrac{2}{13}\displaystyle\int_0^{\sqrt{(13/2)}} \sqrt{(936 - 144x^2)}\,dx$

$$= \frac{24}{13}\int_0^{\sqrt{(13/2)}} \sqrt{\left(\frac{13}{2} - x^2\right)}\,dx.$$

Let $\quad x = \sqrt{(13/2)}\,.\,\sin\theta, \qquad \therefore\ dx = \sqrt{(13/2)}\,.\,\cos\theta\,d\theta.$

When $x = 0$, $\theta = 0$, and when $x = \sqrt{(13/2)}$, $\theta = \frac{1}{2}\pi$.

$$\therefore\ \text{area} = \frac{24}{13}\int_0^{\pi/2} \frac{13}{2}\cos^2\theta\,d\theta = 12\int_0^{\pi/2} \frac{1 + \cos 2\theta}{2}\,d\theta$$

$$= 6\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 3\pi.$$

Odd and Even Functions

5. If $f(x) = f(-x)$, then $f(x)$ is said to be an *even function* of x. Thus x^2, $x^6 + 4x^2 - 3x^4 + 2$, $\cos x$, $\cosh x$, $\sec 2x$, $\sin^2 x$ are all even functions of x (expansions will all contain only even powers of x and hence the name).

If $f(-x) = -f(x)$, then $f(x)$ is said to be an *odd function* of x. Thus x, $\sin x$, $\sinh 2x$, $x - 3x^3 + x^5$, etc., are all odd functions of x (expansions contain only odd powers of x).

Any even function graph is symmetrical about OY, and thus, from graphical inspection, **when $f(x)$ is an even function of x,**

$$\int_{-a}^{+a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

Any odd function between $x = -a$ and $x = +a$ has as much area below the x-axis of the graph as above it, and thus, **when $f(x)$ is an odd function of x,**

$$\int_{-a}^{+a} f(x)\,dx = 0.$$

Solids and Surfaces of Revolution

6. Consider the usual element of area $y\,\delta x$ under a curve $y = f(x)$ between the values $x = a$ and $x = b$, with the corresponding values of $y = c$ and $y = d$.

If the area under this curve be rotated through one complete revolution about OX, the element of volume generated will be that

of a cylindrical disc of radius y and height δx, and will therefore be $\pi y^2 \, \delta x$, where $\delta x \to 0$.

Hence **the total volume generated** $= \int_a^b \pi y^2 \, dx$. Similarly, if the area between the curve and the y-axis be rotated through one complete revolution about OY, **the volume generated will be** $\int_c^d \pi x^2 \, dy$.

If δs be an element of arc of this curve, then the surface area of the elemental disc formed is $2\pi y \, \delta s$, where $\delta s \to 0$.

Therefore the surface area generated in one complete revolution about OX is $\int_{x=a}^{x=b} 2\pi y \, ds$.

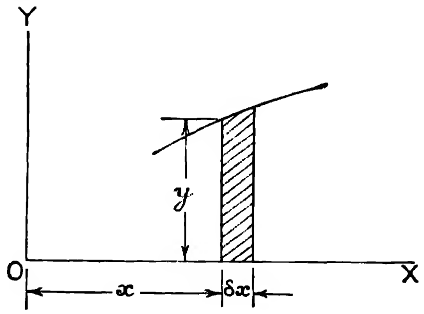
Similarly, for a complete revolution about OY, the surface generated

$$= \int_{x=}^{x=b} 2\pi x \, ds.$$

In these two latter cases, if the equation of the curve be given in the Cartesian form, ds will be replaced in the integral by its equivalent value $\sqrt{\{1 + (dy/dx)^2\}} \, . \, dx$, or $\sqrt{\{1 + (dx/dy)^2\}} \, . \, dy$, whichever is more suitable.

N.B.—In the case of revolution of the area between two curves about one of the axes, it will be necessary to find the difference between the volumes generated by the area under each curve separately, and in all cases a diagram will be of great assistance.

Example 3 (L.U.).—A solid is generated by the revolution of the area under a curve about the axis of y. Show that its volume is $2\pi\int xy \, dx$, the integral being taken between the appropriate limits.



The finite area included between the parabola $y^2 = 4ax$ and the curve $ay^2 = x^3$ is rotated through four right angles about the directrix of the parabola. Find the volume of the solid generated.

Consider the usual element of area $y\,\delta x$, in the diagram, rotated about OY. It will form a cylindrical annulus whose base area is $2\pi x\,\delta x$ and height y, where $\delta x \to 0$, and the volume of the element will therefore equal $2\pi xy\,\delta x$. Hence the total volume generated is the sum of all such elements and equals

$$\operatorname*{Lt}_{\delta x \to 0} \Sigma 2\pi xy\,\delta x = \int 2\pi xy\,dx = 2\pi \int xy\,dx,$$

with appropriate limits.

The directrix of the parabola

$$y^2 = 4ax \quad \ldots\ldots\ldots\ldots \quad \text{(i)}$$

is $x = -a$.

Using the first part of the question with rotation about this line, the volume generated, as obtained by replacing x by $x + a$, is

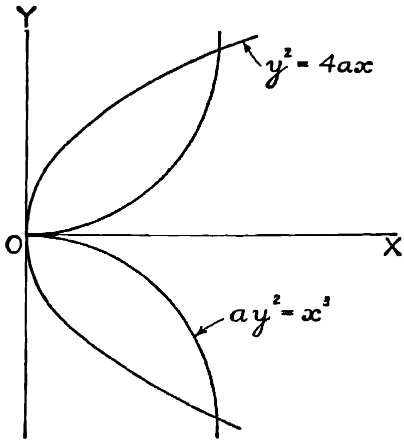
$$2\pi \int (x + a)y\,dx.$$

$$ay^2 = x^3. \qquad \text{(ii)}$$

Substituting from (i) in (ii) for y^2, the points of intersection of the two curves are given by

$$4a^2x = x^3, \quad \text{i.e. } x = 0 \text{ or } \pm 2a$$

($-$ve sign impossible if a positive).



From the diagram, the volume generated by the parabola between the above limits, about the directrix,

$$= 2 \times 2\pi \int_0^{2a} (x + a)y\,dx \qquad \text{(taking } y \text{ +ve)}$$

$$= 4\pi \int_0^{2a} (x + a)\sqrt{(4ax)}\,dx.$$

For the curve (ii) the volume generated in the revolution about the directrix

$$= 2 \times 2\pi \int_0^{2a} (x + a)y\,dx \qquad \text{(taking } y \text{ +ve)}$$

$$= 4\pi \int_0^{2a} (x + a)\sqrt{(x^3/a)}\,dx.$$

$\therefore$ Volume required

$$= 4\pi \int_0^{2a} \left\{(x + a)2a^{1/2}x^{1/2} - (x + a)\frac{x^{3/2}}{a^{1/2}}\right\} dx$$

$$= 4\pi \int_0^{2a} \left\{2a^{1/2}x^{3/2} + 2a^{3/2}x^{1/2} - \frac{x^{5/2}}{a^{1/2}} - x^{3/2}a^{1/2}\right\} dx$$

$$= 4\pi \int_0^{2a} \left\{a^{1/2}x^{3/2} + 2a^{3/2}x^{1/2} - \frac{x^{5/2}}{a^{1/2}}\right\} dx$$

$$= 4\pi \left[a^{1/2}\frac{x^{5/2}}{\frac{5}{2}} + 2a^{3/2}\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{7/2}}{\frac{7}{2}a^{1/2}}\right]_0^{2a}$$

$$= 4\pi \left[\tfrac{2}{5}a^{1/2}4\sqrt{2}\,.\,a^{5/2} + \tfrac{4}{3}a^{3/2}2\sqrt{2}\,.\,a^{3/2} - \tfrac{2}{7}\frac{8\sqrt{2}\,.\,a^{7/2}}{a^{1/2}}\right]$$

$$= 4\pi\,.\,8\sqrt{2}\,.\,a^3[\tfrac{1}{5} + \tfrac{1}{3} - \tfrac{2}{7}]$$

$$= 32\pi \times \tfrac{26}{105}\sqrt{2}\,.\,a^3 = \frac{832\sqrt{2}}{105}\,.\,\pi a^3 \text{ cubic units.}$$

LENGTHS OF CURVES

7. In ascertaining the length of a curve between $x = a$ and $x = b$, the required value is given by $\int_{x=a}^{x=b} ds$, where δs is the element of length of arc.

Using the values for the replacement of ds, **the length of curve**

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

Example 4 (L.U.).—Find (i) the area of the segment, (ii) the length of arc cut off from the parabola $y^2 = 8x$ by the line $4x - y - 4 = 0$.

The parabola is $\qquad y^2 = 8x, \qquad \ldots\ldots\ldots$ (i)

and the equation of the line is $4x - y - 4 = 0. \qquad \ldots\ldots\ldots$ (ii)

Solving (i) and (ii) for the points of intersection of the straight line and parabola,

$$y^2/2 - y - 4 = 0, \text{ i.e. } y^2 - 2y - 8 = 0,$$

from which $\qquad y = 4$ or -2,

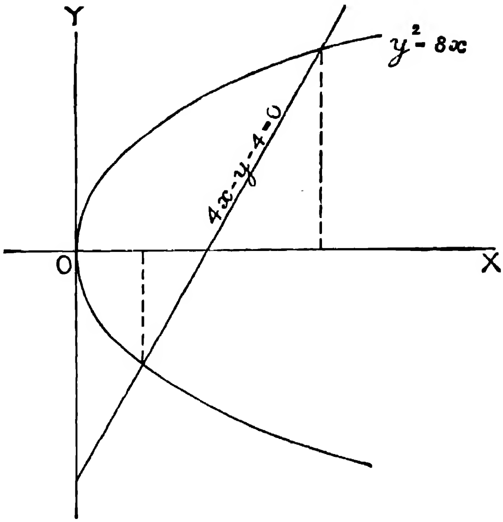
and from (i), $\qquad x = 2$ or $\frac{1}{2}$.

The diagram is a rough representation of the curves (i) and (ii). The equation (i) can be put in the form

$$y = 2\sqrt{(2x)} \qquad \text{(iii)}$$

$$y = -2\sqrt{(2x)}. \qquad \text{(iv)}$$

The equation (iii) represents the top half of the parabola, and the equation (iv) represents the bottom half.



The straight line (i) cuts the x-axis at $x = 1$. From careful consideration of the graph, the area required is the area under the graph of (iii) between $x = 0$ and $x = 2$ added to the positive value of the area under the curve (iv) between $x = 0$ and $x = \frac{1}{2}$, and the total diminished by the area under the line (ii) between $x = \frac{1}{2}$ and $x = 2$.

Hence the required area

$$= \int_0^{2} 2\sqrt{2} \, . \, x^{1/2}\, dx + \left| \int_0^{1/2} 2\sqrt{2} \, . \, x^{1/2}\, dx \right| - \int_{1/2}^{2} 4(x-1)\, dx$$

$$= 2\sqrt{2}\left[\tfrac{2}{3}x^{3/2}\right]_0^2 + 2\sqrt{2}\left[\tfrac{2}{3}x^{3/2}\right]_0^{1/2} - 4\left[\tfrac{1}{2}x^2 - x\right]_{1/2}^2$$

$$= (4\sqrt{2}/3)[2\sqrt{2} + \sqrt{2}/4] - 4[-\tfrac{1}{8} + \tfrac{1}{2}]$$

$$= 6 - \tfrac{3}{2} = \tfrac{9}{2} \text{ square units.}$$

The required length of arc is obtained by using $y^2 = 8x$ and length of arc $= \int \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}\, dy$, where y has the limits -2 and $+4$.

Then $dx/dy = y/4$, and the length of arc

$$S = \int_{-2}^{+4} \sqrt{(1 + y^2/16)}\, dy = \tfrac{1}{4}\int_{-2}^{+4} \sqrt{(16 + y^2)}\, dy$$

$$= \tfrac{1}{4}\left[y\sqrt{(16 + y^2)} - \int \frac{y^2}{\sqrt{(16 + y^2)}}\, dy\right]_{-2}^{+4} \qquad \text{(integration by parts)}$$

$$= \tfrac{1}{4}\left[y\sqrt{(16 + y^2)} - \int \frac{16 + y^2 - 16}{\sqrt{(16 + y^2)}}\, dy\right]_{-2}^{4}$$

$$= \tfrac{1}{4}\left[y\sqrt{(16 + y^2)} - \int \sqrt{(16 + y^2)}\, dy + 16\int \frac{dy}{\sqrt{(16 + y^2)}}\right]_{-2}^{+4}$$

$$= \left[\tfrac{1}{4}y\sqrt{(16 + y^2)} + 4\sinh^{-1} y/4\right]_{-2}^{4} - S.$$

$$\therefore\ 2S = 4\sqrt{2} + 4\sinh^{-1} 1 - [-\tfrac{1}{2}\sqrt{20} + 4\sinh^{-1}(-\tfrac{1}{2})]$$

$$= 4\sqrt{2} + 4\sinh^{-1} 1 + \sqrt{5} + 4\sinh^{-1}\tfrac{1}{2}.$$

$$\therefore\ S = (\tfrac{1}{2})[4\sqrt{2} + \sqrt{5} + 4\sinh^{-1} 1 + 4\sinh^{-1}\tfrac{1}{2}].$$

8. When dealing with the *parametric equation* of a curve $x = f_1(t)$, $y = f_2(t)$, it is known that, with the usual notation,

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}.$$

Therefore the length of curve is given by

$$\int\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\,.\,dx = \int\sqrt{\left\{1 + \left(\frac{\dot{y}}{\dot{x}}\right)^2\right\}}\,.\,\frac{dx}{dt}\,dt$$

$$= \int\sqrt{\left(\frac{\dot{x}^2 + \dot{y}^2}{\dot{x}^2}\right)}\,.\,\dot{x}\,dt = \int\sqrt{(\dot{x}^2 + \dot{y}^2)}\,dt,$$

the integral being taken between the requisite limits.

Example 5.—Find the length of arc from the origin to the point where $t = 1$ on the parabola $x = at^2$, $y = 2at$.

From the equation to the curve

$$\dot{x} = 2at,\ \dot{y} = 2a,$$

and the limits for t are 0 and 1.

Hence length of arc (S) required is given by

$$S = \int_0^1 \sqrt{(\dot{x}^2 + \dot{y}^2)}\, dt = \int_0^1 \sqrt{(4a^2t^2 + 4a^2)}\, dt$$

$$= 2a\int_0^1 \sqrt{(t^2 + 1)}\, dt.$$

Let $$t = \sinh\theta, \quad \therefore\ dt = \cosh\theta\, d\theta,$$

and $$S = 2a\int_{t=0}^{t=1} \sqrt{(\sinh^2\theta + 1)}\,.\,\cosh\theta\, d\theta$$

$$= 2a\int_{t=0}^{t=1} \cosh^2\theta\, d\theta = a\int_{t=0}^{t=1} (1 + \cosh 2\theta)\, d\theta$$

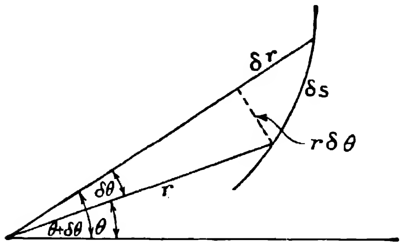
$$= a\Big[\theta + \tfrac{1}{2}\sinh 2\theta\Big]_{t=0}^{t=1} = a\Big[\theta + \sinh\theta\cos\theta\Big]_{t=0}^{t=1}$$

$$= a\Big[\theta + \sinh\theta\ \sqrt{(\sinh^2\theta + 1)}\Big]_{t=0}^{t=1}$$

$$= a\Big[\sinh^{-1} t + t\sqrt{(t^2 + 1)}\Big]_0^1$$

$$= a\Big[\sinh^{-1} 1 + \sqrt{2}\Big]$$

$$= a\Big[\log_e(1 + \sqrt{2}) + \sqrt{2}\Big].$$



9. When dealing with the *polar form* $r = f(\theta)$ of a curve, with the standard notation, it can be seen from the diagram (using Pythagoras' theorem) that $(\delta s)^2 = r^2(\delta\theta)^2 + (\delta r)^2$ in the limit when $\delta\theta \to 0$,

i.e. $$\left(\frac{\delta s}{\delta\theta}\right)^2 = r^2 + \left(\frac{\delta r}{\delta\theta}\right)^2, \text{ as } \delta\theta \to 0,$$

i.e. $$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2, \quad \therefore\ \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}.$$

Hence length of arc $$= \int ds = \int \frac{ds}{d\theta}\,.\,d\theta$$

$$= \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

CENTRES OF GRAVITY, CENTROIDS, ETC.

10. Definition.—The centre of gravity of a body is the point through which the resultant weight acts.

Choosing the usual three mutually perpendicular axes OX, OY, OZ and (x, y, z) as the co-ordinates of a particle of weight δw of the body, and $(\bar{x}, \bar{y}, \bar{z})$ as the co-ordinates of the centre of gravity of the body, then, by taking moments about the plane YOZ,

$$W\bar{x} = \operatorname*{Lt}_{\delta w \to 0} \Sigma x \, \delta w,$$ where W is the total weight of the body.

Hence $$W\bar{x} = \int x \, dw,$$

i.e. $$\bar{x} = \frac{\int x \, dw}{W} = \frac{\int x \, dw}{\int dw}.$$

Similarly, $$\left.\begin{aligned} \bar{x} &= \frac{\int x \, dw}{W} = \frac{\int x \, dw}{\int dw}. \\ \bar{y} &= \frac{\int y \, dw}{W} = \frac{\int y \, dw}{\int dw}; \\ \bar{z} &= \frac{\int z \, dw}{W} = \frac{\int z \, dw}{\int dw}, \end{aligned}\right\}$$

the integrations being taken over the whole body.

Definition.—The centre of mass of a body is the point through which the resultant mass acts, and, since weight is proportional to mass, the centre of mass is the same point as the centre of gravity.

Applying these results to lengths, areas, and volumes, and replacing the element of weight (δw) by the element of length (δs), the element of area (δA), and the element of volume (δV) in the respective cases, the point arrived at is known as the *centroid* of the body. The centroid will be the same as the centre of gravity, providing that the body be uniform, which is generally the case.

In practically all problems to be considered, there is at least one axis of symmetry, along which the centroid must lie, and hence only two at the most of the previous formulæ will generally be required.

11. The following are well-known centres of gravity (centroids) of *uniform* bodies.

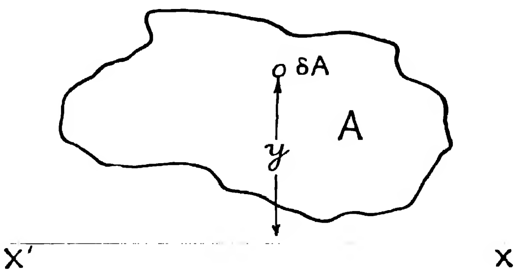
Body	Centre of Gravity
Rod.	Mid-point.
Rectangle.	Intersection of diagonals.
Triangular lamina.	Intersection of medians.
Curved surface of right circular cone.	$\frac{1}{3}$ way up the axis from the centre of the base.
Solid right circular cone.	$\frac{1}{4}$ way up axis from centre of base.
Pyramid.	$\frac{1}{4}$ way up the join of the centroid of the base to the vertex.
Solid or curved surface of cylinder.	Mid-point of axis.
Curved surface of hemisphere.	Mid-point of radius perpendicular to the base.
Solid hemisphere.	$\frac{3}{8}$ along radius perpendicular to base.

12. Pappus' Theorems.

These are theorems suitable for finding the centroid of a closed area or a curve by considering the solids of revolution about a certain axis, and are as follows:

13. Theorem I.—If a closed area be rotated through any angle not greater than 2π radians about a line in its plane that does not intersect it, the volume generated is equal to the product of the area and the length of the path of the centroid.

Consider a closed plane area A and a line XX′ not intersecting it. δA is an element of this area at a distance y from XX′, and $\bar{y}$ is the distance of the centroid of the area A from XX′.



If the plane area A be rotated through an angle θ radians about XX′, the element of volume generated is $\theta y\,\delta A$.

Hence the total volume generated

$$= \underset{\delta A \to 0}{\mathrm{Lt}}\ \Sigma \theta y\,\delta A$$

$$= \theta \underset{\delta A \to 0}{\mathrm{Lt}}\ \Sigma y\,\delta A \qquad (\theta \text{ constant}).$$

But $$\operatorname*{Lt}_{\delta A \to 0} \Sigma y\, \delta A = \int y\, dA = A\bar{y}.$$

Therefore total volume generated $= \theta\bar{y}A$.

But $\theta\bar{y} =$ distance travelled by the centroid in the rotation. Therefore total volume generated is equal to the product of the area and the length of path traced out by the centroid.

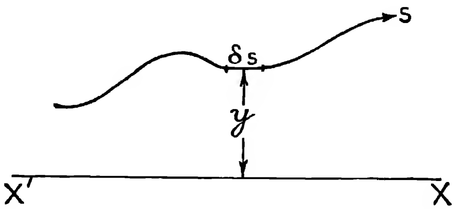
N.B.—(i) If $\theta > 2\pi$, the volume generated after $\theta > 2\pi$ will merge into the volume generated when $\theta < 2\pi$.

(ii) XX′ must not intersect the area A, or the part of the volume generated by the area below XX′ will become part of that generated by the area above XX′, when $\theta > \pi$. (A straight line forming part of the area A is not considered as intersecting it.)

(iii) In most problems, a complete revolution is considered, i.e. $\theta = 2\pi$.

14. Theorem II.—If a plane curve be rotated through an angle, not greater than 2π radians, about a line in its plane not intersecting it, the surface generated is equal to the product of the length of the curve and the length of the path of the centroid.

Consider a plane curve, of length s, not intersecting a line XX′ in its plane. δs is an element of length of the curve at a distance y from XX′, and $\bar{y}$ is the distance of the centroid of the curve from XX′.



If the curve be rotated through an angle θ radians ($\theta \not> 2\pi$) about XX′, the surface generated by the element is $\theta y\, \delta s$.

Thus the total surface area generated

$$= \operatorname*{Lt}_{\delta s \to 0} \Sigma\theta y\, \delta s = \theta \operatorname*{Lt}_{\delta s \to 0} \Sigma y\, \delta s.$$

But $$\operatorname{Lt} \Sigma y\, \delta s = \int y\, ds = \bar{y}s.$$

$\therefore$ Surface area $= \theta\bar{y}s =$ length of path of centroid $\times$ length of curve.

The notes are similar to those on Theorem I.

Example 6 (L.U.).—Trace roughly the curve $xy^2 = 4(2 - x)$. Find the area enclosed by the curve and the y-axis, also the volume of the solid formed by a revolution of the curve through four right angles about the y-axis.

Hence determine the position of the centroid of the area between the curve and the y-axis.

The equation of the curve can be put in the form

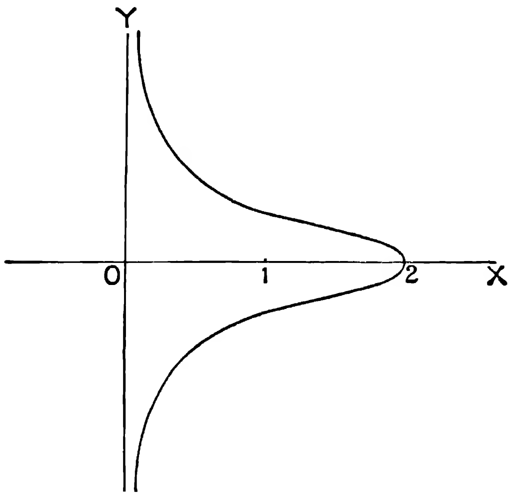
$$y^2 = 4(2 - x)/x.$$

From this it can be seen that the curve is symmetrical about the x-axis and lies entirely between $x = 0$ and $x = 2$.

The equation can also be written

$$y^2 = 8/x - 4. \qquad \text{(i)}$$

$$\therefore\ 2y\frac{dy}{dx} = -8/x^2.$$



Hence $dy/dx = 0$, when $x = \infty$, which is impossible, therefore there are no maxima or minima.

When $x = 0$, $\qquad y = \pm\infty$.

When $x = 2$, $\qquad y = 0$, and $\dfrac{dy}{dx} = \infty$.

When $x = 1$, $\qquad y = \pm 2$.

From all these results the rough graph shown is obtained.

The area enclosed between the curve and the y-axis $= 2\int_0^x$

From equation (i), $$x = \frac{8}{y^2 + 4}.$$

$$\therefore \text{ Area required} = 16\int_0^\infty \frac{dy}{y^2+4} = 8\left[\tan^{-1}\tfrac{1}{2}y\right]_0^\infty$$
$$= 4\pi.$$

The volume V of the solid formed by the revolution of the curve about OY

$$= 2\int_0^\infty \pi x^2\,dy = \int_0^\infty \frac{128\pi}{(y^2+4)^2}\,dy = 128\pi\int_0^\infty \frac{dy}{(y^2+4)^2}.$$

Let $$y = 2\tan\theta, \quad \therefore\ dy = 2\sec^2\theta\,d\theta.$$

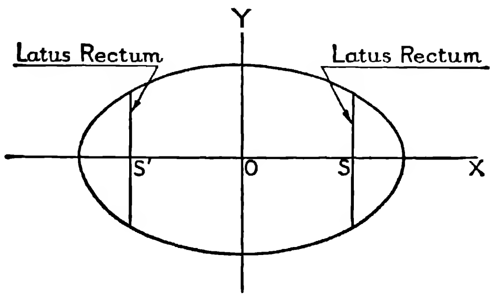
When $y = 0$, $\theta = 0$; when $y = \infty$, $\theta = \pi/2$.

$$\therefore\ V = 128\pi\int_0^{\pi/2} \frac{2\sec^2\theta\,d\theta}{16\sec^4\theta} = 16\pi\int_0^{\pi/2}\cos^2\theta\,d\theta$$
$$= 8\pi\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta = 8\pi\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$
$$= 4\pi^2.$$

By symmetry, the centroid will lie on OX, and if $\bar{x}$ be its distance from OY, then, by Pappus' theorem,

$$4\pi^2 = 4\pi\,.\,2\pi\bar{x}.$$
$$\therefore\ \bar{x} = \tfrac{1}{2}.$$

Example 7 (L.U.).—A smaller segment of the ellipse $3x^2 + 4y^2 = 1$, cut off by one latus rectum, is rotated about the other latus rectum through four right angles. Show that the volume of the solid thus formed $= \pi(4\pi + 3\sqrt{3})/72$.



The equation of the ellipse can be written

$$\frac{x^2}{\frac{1}{3}} + \frac{y^2}{\frac{1}{4}} = 1. \qquad \text{(i)}$$

If a be the semi-major axis, b the semi-minor axis, and e the eccentricity of the ellipse,

$$a^2 = \tfrac{1}{3}, \quad b^2 = \tfrac{1}{4},$$
$$\tfrac{1}{4} = \tfrac{1}{3}(1 - e^2), \quad \text{i.e.} \quad e^2 = \tfrac{1}{4}.$$
$$\therefore\ e = \tfrac{1}{2}.$$

If S and S′ be the foci of the ellipse, O being its centre, then

$$\text{OS} = ae = \frac{1}{\sqrt{3}} \cdot \frac{1}{2} = \frac{1}{2\sqrt{3}}.$$

Hence, considering the area of the segment at the right of OY formed by the corresponding latus rectum and the ellipse, its area

$$A = 2\int_{1/(2\sqrt{3})(y+\text{ve})}^{1/\sqrt{3}} y\,dx = 2\int_{1/(2\sqrt{3})}^{1/\sqrt{3}} \tfrac{1}{2}\sqrt{(1-3x^2)}\,dx = \sqrt{3}\int_{1/(2\sqrt{3})}^{1/\sqrt{3}} \sqrt{(\tfrac{1}{3}-x^2)}\,dx.$$

Let $$x = \frac{1}{\sqrt{3}}\sin\theta, \quad \therefore\ dx = \frac{1}{\sqrt{3}}\cos\theta\,d\theta.$$

When $x = 1/\sqrt{3}$, $\theta = \frac{1}{2}\pi$; when $x = 1/(2\sqrt{3})$, $\theta = \frac{1}{6}\pi$.

$$\therefore\ A = \sqrt{3}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} \sqrt{(\tfrac{1}{3}-\tfrac{1}{3}\sin^2\theta)}\,\frac{1}{\sqrt{3}}\cos\theta\,d\theta$$

$$= \frac{1}{\sqrt{3}}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}\cos^2\theta\,d\theta = \frac{1}{2\sqrt{3}}\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2\sqrt{3}}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} = \frac{1}{2\sqrt{3}}\left[\tfrac{1}{2}\pi - \tfrac{1}{6}\pi - \tfrac{1}{4}\sqrt{3}\right]$$

$$= \frac{1}{2\sqrt{3}}\left[\tfrac{1}{3}\pi - \tfrac{1}{4}\sqrt{3}\right].$$

The moment of this area A about OY

$$= 2\int_{1/(2\sqrt{3})}^{1/\sqrt{3}} xy\,dx \qquad (y+\text{ve})$$

$$= 2\int_{1/(2\sqrt{3})}^{1/\sqrt{3}} \tfrac{1}{2}x\sqrt{(1-3x^2)}\,dx = \int_{1/(2\sqrt{3})}^{1/\sqrt{3}} x\sqrt{(1-3x^2)}\,dx$$

$$= \left[-\tfrac{1}{9}(1-3x^2)^{3/2}\right]_{1/(2\sqrt{3})}^{1/\sqrt{3}} = \tfrac{1}{9}(\tfrac{3}{4})^{3/2} = \frac{1}{8\sqrt{3}}.$$

If $\bar{x}$ be the distance of the centroid of the segment from OY, then

$$\bar{x} = \frac{\text{Total moment about OY}}{\text{Total area}}$$

$$= \frac{1/(8\sqrt{3})}{1/(2\sqrt{3})[\frac{1}{3}\pi - \frac{1}{4}\sqrt{3}]} = \frac{1}{4(\frac{1}{3}\pi - \frac{1}{4}\sqrt{3})}.$$

Thus the distance of the centroid from the other latus rectum

$$= \frac{1}{2\sqrt{3}} + \frac{1}{4(\frac{1}{3}\pi - \frac{1}{4}\sqrt{3})} = \frac{\frac{1}{3}.4\pi - \sqrt{3} + 2\sqrt{3}}{8\sqrt{3}(\frac{1}{3}\pi - \frac{1}{4}\sqrt{3})}$$

$$= \frac{4\pi + 3\sqrt{3}}{24\sqrt{3}(\frac{1}{3}\pi - \frac{1}{4}\sqrt{3})}.$$

Using Pappus' theorem and V for the required volume,

$$V = 2\pi\left[\frac{4\pi + 3\sqrt{3}}{24\sqrt{3}(\frac{1}{3}\pi - \frac{1}{4}\sqrt{3})}\right] \times \left[\frac{1}{2\sqrt{3}}(\tfrac{1}{3}\pi - \tfrac{1}{4}\sqrt{3})\right]$$

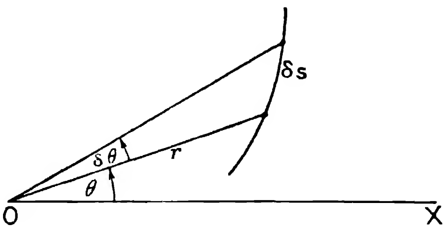
$$= \frac{\pi(4\pi + 3\sqrt{3})}{72}.$$

Example 8 (L.U.).—A sector of a plane curve whose equation in polar co-ordinates is of the form $r = f(\theta)$ is rotated about the initial line. Show that the volume of the solid so generated can be expressed in terms of the integral

$$\tfrac{2}{3}\pi \int r^3 \sin\theta \, d\theta.$$

The curve $r = a(1 + \cos\theta)$ is rotated through an angle of π radians about the initial line. Find the superficial area and the volume of the solid so generated.

Using standard notation as shown in the diagram, with $\delta\theta \to 0$, the area of the element of sector shown is $\frac{1}{2}r^2\,\delta\theta$, to the first order of small quantities. This



elemental sector can be considered as a triangle, and the centre of gravity is two-thirds of the way up the median from O, i.e. $\frac{2}{3}r\sin\theta$ from the initial line OX, to the first order of small quantities. Hence, in a complete revolution about OX, the length of path of the centre of gravity is

$$2\pi \times \tfrac{2}{3}r\sin\theta.$$

Using the theorem of Pappus, the volume generated by this element in the revolution about OX

$$= \tfrac{1}{2}r^2\,\delta\theta \times 2\pi \times \tfrac{2}{3}r\sin\theta$$

$$= \tfrac{2}{3}\pi r^3 \sin\theta\,\delta\theta.$$

Total volume generated by the sector is the sum of all such elements as $\delta\theta \to 0$, and is equal to

$$\operatorname*{Lt}_{\delta\theta\to 0} \Sigma\tfrac{2}{3}\pi r^3 \sin\theta\,\delta\theta = \tfrac{2}{3}\pi\int r^3 \sin\theta\, d\theta.$$

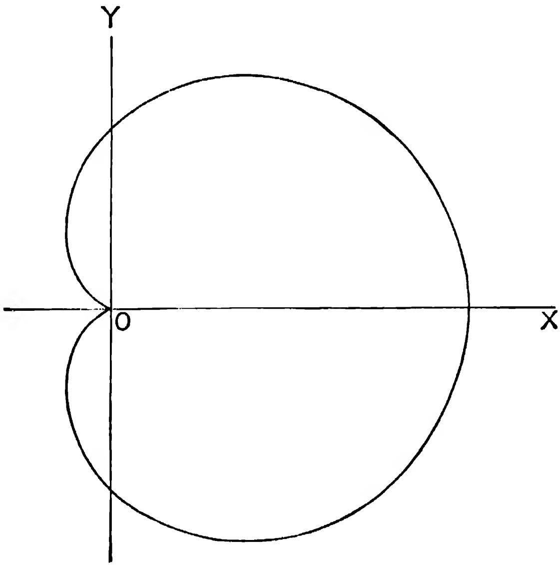
The diagram gives a rough idea of the shape of the curve $r = a(1 + \cos\theta)$ with a positive. It is known as the cardioid.

A rotation of the whole loop about OX through π radians is equivalent to the rotation of the upper half about OX through 2π radians, therefore limits for θ are 0 and π and, using the above formula, the volume generated

$$= \tfrac{2}{3}\pi \int_0^\pi r^3 \sin\theta \, d\theta = \tfrac{2}{3}\pi a^3 \int_0^\pi (1 + \cos\theta)^3 \sin\theta \, d\theta$$

$$= \tfrac{2}{3}\pi a^3 \int_0^\pi -(1 + \cos\theta)^3 \, d(1 + \cos\theta) = -\tfrac{2}{3}\pi a^3 \Big[\tfrac{1}{4}(1 + \cos\theta)^4\Big]_0^\pi$$

$$= -\tfrac{2}{3}\pi a^3 \Big[0 - \tfrac{1}{4} \, . \, 2^4\Big] = \tfrac{1}{3}(8\pi a^3).$$



From the first diagram, the superficial area generated by the element δs is $2\pi \, . \, r \sin\theta \, . \, \delta s$.

$\therefore$ Total area generated $= \int 2\pi r \sin\theta \, ds$

$$= \int 2\pi r \sin\theta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} \, d\theta.$$

But $\quad dr/d\theta = -a \sin\theta.$

$$\therefore \text{ Required area} = \int_0^\pi 2\pi r \sin\theta \sqrt{(r^2 + a^2\sin^2\theta)}\, d\theta$$

$$= \int_0^\pi 2\pi a \sin\theta(1+\cos\theta)\sqrt{\{a^2(1+\cos\theta)^2 + a^2\sin^2\theta\}}\, d\theta$$

$$= 2\pi a^2 \int_0^\pi \sin\theta(1+\cos\theta)\sqrt{(2+2\cos\theta)}\, d\theta$$

$$= 2\sqrt{2}\,.\,\pi a^2 \int_0^\pi \sin\theta(1+\cos\theta)^{3/2}\, d\theta$$

$$= -2\sqrt{2}\,.\,\pi a^2 \int_0^\pi (1+\cos\theta)^{3/2}\,.\,d(1+\cos\theta)$$

$$= -2\sqrt{2}\,.\,\pi a^2 \left[\frac{(1+\cos\theta)^{5/2}}{5/2}\right]_0^\pi$$

$$= -(4\sqrt{2}/5)\,.\,\pi a^2[0 - 2^{5/2}]$$

$$= (32/5)\pi a^2.$$

EXAMPLES ON CHAPTER XIV

The following questions are all taken from London University examination papers.

1. Sketch roughly the curve whose equation is $ay^2 = (x-a)^2(4a-x)$, and find the area of the loop.

2. Tangents OP, OQ are drawn from the origin O to the parabola $x^2 - 2y + 4 = 0$. Show that the area enclosed by the parabola and these tangents is 8/3 square units, and find the arcual distance of P or Q from the vertex.

3. If the co-ordinates of a point on the curve $3ay^2 = x(a-x)^2$ are expressed in terms of a parameter in the form

$$x = 3at^2,\quad y = a(t - 3t^3),$$

show how the curve is traced out as t increases from $-\infty$ to $+\infty$.

Show that the length of the arc of this curve from the origin to the point of the loop where the tangent to the curve makes an angle ψ with the y-axis is

$$(a/9)[3\tan\tfrac{1}{2}\psi + \tan^3\tfrac{1}{2}\psi).$$

4. The co-ordinates of a point on a plane curve are given parametrically by $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$. If s be the length of the arc of the curve from the point given by $t = 0$, to the point where the tangent makes an angle ψ with the tangent at the point $t = 0$, show that $s = 16a\sin^2\psi/6$.

5. State and prove a formula for the area of a curve whose equation is given in polar co-ordinates.

Sketch the curve whose equation is $r = a(\sqrt{2}\,.\cos\theta - 1)$, and show that the smaller loop has an area of $(\pi - 3)\frac{1}{2}a^2$. Find also the area of the larger loop.

6. Show that $\displaystyle\int \frac{a^2\,dx}{(a+x)(x^2+2a^2)} = \frac{1}{3\sqrt{2}}\tan^{-1}\frac{x}{a\sqrt{2}} + \frac{1}{6}\log_e\frac{(x+a)^2}{x^2+2a^2}$.

Prove that the area in the positive quadrant, bounded by the curve $y(x+a)(x^2+2a^2) = a^4$ and the axes of co-ordinates, is $(\log_e 2 + \pi/\sqrt{2})a^2/6$.

7. Sketch the form of the plane curve whose equation is $8ay^2 = x(2a-x)^2$. Find the area of its loop and show that, if the tangent at any point P of the curve meet the axis of y at the point T, and O is the origin, then OP = 2OT.

8. Sketch roughly the curve whose equation is $y^2 = x^3/(a-x)$, where a is positive, and find the area included between the curve and its asymptote.

Find also the volume generated on revolving the curve about its asymptote.

9. A plane curve touches the axis of x at O, the origin. The tangent at a point P of the curve meets the axis of x at T and makes with it an angle ψ. If the ordinate of P is $a\sin^4\psi$, show that the length of the arc OP is 4PT/3.

Show also that, if the perpendicular to the axis of x at T meets the normal at P at the point G, the radius of curvature at P is 4PG.

10. Find the whole length of the curve given by the equations $x = a\cos^3\theta$, $y = a\sin^3\theta$.

Determine also the surface of the solid formed by revolving the part of the curve which lies on one side of the x-axis about the x-axis.

11. A circle of radius R rolls on the outside of a fixed circle of radius $3R$. Taking the centre of the fixed circle as origin, and the line joining the initial positions of the centres as axis of x, show that the co-ordinates of the point P of the rolling circle, which was initially in contact with the fixed circle, are

$$x = 4R\cos\theta - R\cos 4\theta, \quad y = 4R\sin\theta - R\sin 4\theta,$$

where θ is the angle turned through by the line of centres. Show that the length of the curve traced out by P as θ increases from 0 to 2π is $32R$.

12. A circle of radius a rolls without slipping on the outside of an equal circle. Show that the path of a point on the circumference of the rolling circle is given by the equations

$$x = a(2\cos\phi - \cos 2\phi), \quad y = a(2\sin\phi - \sin 2\phi),$$

the origin being at the centre of the fixed circle, and the axis of x passing through the position P of the tracing point at which it is on the fixed circle.

Show also that, by taking P as pole and the x-axis as initial line, the curve may be expressed in polar co-ordinates by the equation $r = 2a(1-\cos\theta)$, and prove that it encloses an area equal to six times that of either circle.

13. Sketch roughly the curve $y^2(2a-x) = x^3$, and find the volume generated by the revolution of the curve about its asymptote through four right angles.

14. The area bounded by the arc of a plane curve, the x-axis, and a straight line through the origin, is rotated through four right angles about OX. By means of Pappus' theorem, or otherwise, show that the volume of the solid generated is

$$\frac{2\pi}{3}\int\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)y\,dt,$$

taken between the proper limits, where t is any parameter, in terms of which the co-ordinates of a point on the curve can be expressed.

The area bounded by the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, the x-axis from the origin to the vertex $(a, 0)$, and the radius vector from the origin to the point whose eccentric angle is α, is rotated through four right angles about the x-axis. Show that the volume of the solid generated is $\frac{2}{3}\pi ab^2(1 - \cos\alpha)$.

15. Show that the curve $r = 1 + 2\cos\theta$ consists of an outer and inner loop. If the area of the inner loop be rotated through two right angles about the initial line, show that the volume of the solid thus formed is $\pi/12$.

16. The area bounded by the two parabolas $y^2 = ax$ and $x^2 = ay$ is rotated through 2π radians about the x-axis. Find the volume and the superficial area of the solid so formed.

17. The ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is divided into two parts by the line $2x = a$, and the smaller part is rotated through four right angles about this line. Prove that the volume generated is $\pi(3\sqrt{3}/4 - \pi/3)a^2b$.

18. Show that the volume of the solid obtained by rotating a sector of a plane curve, whose equation in polar co-ordinates is $r = f(\theta)$, through four right angles about the initial line is $\frac{2}{3}\pi\int r^3 \sin\theta\, d\theta$, taken between the appropriate limits.

A solid is formed by rotating the area between the two loops of the curve $r = a(1 + 2\cos\theta)$ through four right angles about the initial line. Find its volume.

19. A cask is generated by rotating an arc of a parabola about a line parallel to the tangent at the vertex. The radius of either end is a, of the middle section b, and the height of the cask is h. Find the area of a section by a plane through the axis, and show that its volume is $\pi h(3a^2 + 4ab + 8b^2)/15$.

20. The area in the first quadrant bounded by the axes and the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ is rotated through four right angles about the x-axis. Show that the area of the surface generated by the curve is $6\pi a^2/5$, and find the volume contained within this surface.

21. Show that the volume of a spherical cap of height h, and of which the radius of the rim is R, is $\frac{1}{6}\pi h(3R^2 + h^2)$.

A double convex lens has diameter $2R$ and central thickness H. Show that the volume is approximately $\frac{1}{2}\pi R^2H$, whatever the ratio of the radii of curvature of the two faces.

22. A spindle is formed by revolving the arc of the parabola $y^2 = 4ax$ cut off by the latus rectum about the directrix. Find its volume.

23. Find the co-ordinates of the centroid of a quadrant of the ellipse

$$x^2/a^2 + y^2/b^2 = 1.$$

If the segment of this ellipse, cut off by a chord joining the extremities of the major and minor axes, be rotated through four right angles about that chord, find the volume of the solid formed.

24. A surface is formed by the revolution of the curve $y = ax^2 + 2bx + c$ about the x-axis. Show that the volume cut off between the planes $x = \pm h$ is equal to $\frac{1}{3}h(A_1 + 4A_2 + A_3)$, where A_1, A_2, A_3 are the areas of the sections by the planes $x = -h$, $x = 0$, $x = h$ respectively.

Show also that the abscissa of the centroid of this volume is

$$\frac{(A_3 - A_1)h}{A_1 + 4A_2 + A_3}.$$

25. Find the area included between the parabola $y^2 = x$ and the straight line $x - 5y + 6 = 0$.

Determine the co-ordinates of the centroid of this area.

26. Find the position of the centroid of an arc of a circle. Using this result, or otherwise, find the position of the centroid of a semicircular area.

A semicircle ABC, radius a, revolves about a line parallel to, and at a distance b from the bounding diameter AC, so that the point B of the circumference of the area is at a distance $(a + b)$ from the axis of rotation. Find (i) the area of the surface, (ii) the volume of the solid formed.

27. Prove the theorem of Pappus relating to the surface obtained by rotating a curve about an axis in its own plane.

A circle of radius a is rotated about an axis in its own plane, distant b ($>a$) from its centre, so as to form a closed ring. The surface of the ring is divided into two parts by a circular cylinder of radius $b - a\cos\theta$ coaxial with the ring. Show that the areas of the two parts are

$$4a\pi[b\theta - a\sin\theta] \text{ and } 4a\pi[b(\pi - \theta) + a\sin\theta].$$

28. Show that the length of one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is $8a$.

Find the centroid of a uniform wire in the form of one arch of this curve, and deduce or prove otherwise that, if an arch of the cycloid be revolved through four right angles about the axis of x, the area of the surface traced out is $64\pi a^2/3$.

29. A hemisphere of radius R has a cylindrical hole of radius a drilled through it, the axis of the hole being along the radius normal to the plane face of the hemisphere. Find the distance of the mass centre of the remainder from the plane face.

CHAPTER XV

First-order Differential Equations

1. Formation of differential equations.

Consider the relation $f(x, y) = 0$ involving n arbitrary constants A, B, etc. By successive differentiation of this with respect to x a further n relationships involving x, y, and the first n derivatives of y with respect to x, as well as some or all of the n arbitrary constants, will be obtained.

Including the original equation $f(x, y) = 0$, there will be $(n + 1)$ relationships from which the constants A, B, etc., can be eliminated, and the resulting relationship will involve d^ny/dx^n, and differential coefficients of lower orders, together with x and y, but no arbitrary constants. Such a relation is known as a *differential equation of the nth order*, and since all the derivatives are with respect to a single independent variable x, the equation is known as an *ordinary differential equation.*

Example 1.—Form the differential equation not containing A, from the relation $y^2 = 4Ax$.

$$y^2 = 4Ax. \qquad \text{(i)}$$

From (i),

$$2y\frac{dy}{dx} = 4A$$

$$\frac{y^2}{x} \qquad \text{[using (i)]}.$$

Therefore, since $y \neq 0$, the required equation is

$$\frac{dy}{dx} = \frac{y}{2x}. \qquad \text{(ii)}$$

The equation (i) is a family of parabolas, whose vertices are at the origin, and whose foci are along the x-axis.

The equation (ii) gives the property of these parabolas, namely, that the slope of the tangent at any point P is equal to the ordinate of P divided by twice its abscissa. From this it can readily be seen that the subtangent at P is bisected at O.

A. Variables separable

5. I (*a*). *y absent* in P and Q.

P and Q will be functions of x only, and the original equation

$$P\frac{dy}{dx}+Q=0$$

becomes
$$\frac{dy}{dx}=-\frac{Q}{P}=F(x).$$

$$\therefore\ y=\int F(x)\,dx+C.$$

I (*b*). *x also absent* in P and Q.

In this case P and Q are constants and the differential equation becomes $dy/dx=k$, where $k=-Q/P=$ constant.

The solution is
$$y=kx+C.$$

II. *x absent* in P and Q.

In this case the differential equation (1) becomes

$$\frac{P}{Q}\frac{dy}{dx}=-1,\quad \text{i.e. } F(y)\frac{dy}{dx}=-1\qquad \left(\frac{P}{Q}=F(y)\right),$$

i.e.
$$\frac{dx}{dy}=-F(y),$$

$$\therefore\ x=-\int F(y)\,dy+C.$$

III. *Both x and y present* in P and Q, but the variables separable.

Taking $P/Q=f(y)/\phi(x)$, the equation (1) can be written

$$f(y)\frac{dy}{dx}+\phi(x)=0.$$

Integrating this with respect to x,

$$\int f(y)\frac{dy}{dx}\,dx+\int\phi(x)\,dx=C.$$

Comparing the first integral with integration by substitution, this equation becomes

$$\int f(y)\,dy+\int\phi(x)\,dx=C.$$

Example 4.—(y absent.)

Solve the equation $(1 + 2x)\frac{dy}{dx} = 1 - x$.

The equation can be written

$$\frac{dy}{dx} = \frac{1 - x}{1 + 2x}$$

$$= \frac{-(1 + 2x)}{2(1 + 2x)} + \frac{3}{2(1 + 2x)}$$

$$= -\frac{1}{2} + \frac{3}{2}\frac{1}{1 + 2x}.$$

Hence by integration the solution is

$$y = \tfrac{3}{4}\log_e(1 + 2x) - \tfrac{1}{2}x + a.$$

Example 5.—(x absent.)

Solve the equation $(1 - y)\frac{dy}{dx} = 1 + y^2$.

The given equation can be written

$$\frac{1 - y}{1 + y^2}\frac{dy}{dx} = 1,$$

i.e.
$$\frac{dx}{dy} = \frac{1 - y}{1 + y^2} = \frac{1}{1 + y^2} - \frac{y}{1 + y^2}.$$

Integrating with respect to y,

$$x = \tan^{-1} y - \tfrac{1}{2}\log_e(1 + y^2) + a.$$

Example 6.—(Both x and y present.)

Solve the equations

(a) $x^2(1 + y)\frac{dy}{dx} + (1 - x)y^2 = 0$,

(b) $x\frac{dy}{dx} + \cot y = 0$, given $y = \pi/4$ when $x = \sqrt{2}$,

(c) $(1 + x^2)\frac{dy}{dx} + (1 - y^2) = 0$.

(a) The equation can be written

$$\frac{1 + y}{y^2}\frac{dy}{dx} + \frac{1 - x}{x^2} = 0,$$

i.e.
$$\left(\frac{1}{y^2} + \frac{1}{y}\right)\frac{dy}{dx} + \frac{1}{x^2} - \frac{1}{x} = 0. \qquad \text{(i)}$$

When P and Q are both constants the equation degenerates to type A, I, and can be readily solved.

When P and Q are not both constants the method of solution is as follows:

Multiply equation (8) by v, where v is an arbitrary function of x to be determined later, containing no arbitrary constant, and is known as the *integrating factor* of equation (8). Then equation (8) becomes

$$v\frac{dy}{dx} + Pvy = vQ. \qquad \qquad (9)$$

Now $$\frac{d}{dx}(vy) = v\frac{dy}{dx} + y\frac{dv}{dx},$$

i.e. $$v\frac{dy}{dx} = \frac{d}{dx}(vy) - y\frac{dv}{dx}.$$

Using this in equation (9), it becomes

$$\frac{d}{dx}(vy) - y\frac{dv}{dx} + Pvy = vQ,$$

i.e. $$\frac{d}{dx}(vy) + y\left(Pv - \frac{dv}{dx}\right) = vQ. \qquad \qquad (10)$$

Now choose v so that the coefficient of y in equation (10) becomes zero,

i.e. $$-\frac{dv}{dx} + Pv = 0.$$

i.e. $$\frac{1}{v}\frac{dv}{dx} = P.$$

Integrating this with respect to x, with no constant of integration,

$$\log_e v = \int P\,dx,$$

$$\therefore\ v = e^{\int P\,dx}.$$

Using this integrating factor $e^{\int P\,dx}$ in equation (10), it becomes

$$\frac{d}{dx}\left(e^{\int P\,dx}y\right) = Qe^{\int P\,dx},$$

and integrating this with respect to x,

$$ye^{\int P\,dx} = \int\left(Qe^{\int P\,dx}\right)dx + C,$$

i.e. $$y = e^{-\int P\,dx}\left\{\int\left(Qe^{\int P\,dx}\right)dx + C\right\}.$$

Example 11.—Solve the equation

$$x\frac{dy}{dx} - 3y = x - 1,$$

given $y = 0$ when $x = 1$.

The given equation can be written as

$$\frac{dy}{dx} - \frac{3}{x}y = 1 - \frac{1}{x}. \qquad \text{(i)}$$

The integrating factor of equation (i) is

$$e^{\int -(3/x)\,dx} = e^{-3\log_e x} = e^{\log_e(1/x^3)} = 1/x^3.$$

Multiplying through the equation (i) by the integrating factor $1/x^3$, it becomes

$$\frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}\,.\,y = \frac{1}{x^3} - \frac{1}{x^4},$$

i.e.

$$\frac{d}{dx}\left(\frac{y}{x^3}\right) = \frac{1}{x^3} - \frac{1}{x^4}.$$

Integrating with respect to x,

$$\frac{y}{x^3} = C - \frac{1}{2x^2} + \frac{1}{3x^3},$$

i.e.

$$y = Cx^3 - \tfrac{1}{2}x + \tfrac{1}{3}.$$

Now $y = 0$ when $x = 1$,

$$\therefore\ 0 = C - \tfrac{1}{2} + \tfrac{1}{3},\quad \therefore\ C = \tfrac{1}{6},$$

i.e.

$$y = \tfrac{1}{6}x^3 - \tfrac{1}{2}x + \tfrac{1}{3}.$$

Example 12 (L.U.).—Solve the equation

$$(1 + x^2)^{5/2}\,dy/dx + x(1 + x^2)^{3/2}\,y = x^2.$$

The given equation can be written as

$$\frac{dy}{dx} + \frac{x}{1 + x^2}\,y = \frac{x^2}{(1 + x^2)^{5/2}}. \qquad \text{(i)}$$

The integrating factor of equation (i) is

$$e^{\int x/(1 + x^2)\,dx} = e^{\frac{1}{2}\log_e(1 + x^2)} = e^{\log_e(1 + x^2)^{\frac{1}{2}}} = (1 + x^2)^{\frac{1}{2}}.$$

Multiplying through equation (i) by this integrating factor $(1 + x^2)^{\frac{1}{2}}$,

$$(1 + x^2)^{\frac{1}{2}}\frac{dy}{dx} + \frac{x}{(1 + x^2)^{\frac{1}{2}}}\,y = \frac{x^2}{(1 + x^2)^2},$$

$$\frac{d}{dx}\{y(1 + x^2)^{\frac{1}{2}}\} = \frac{x^2}{(1 + x^2)^2}.$$

Integrating this with respect to x,

$$y(1+x^2)^{\frac{1}{2}} = \int \frac{x^2}{(1+x^2)^2}\,dx + C = I + C,$$

where
$$I = \int \frac{x^2}{(1+x^2)^2}\,dx.$$

Let
$$x = \tan\theta, \quad \therefore\ dx = \sec^2\theta\,d\theta,$$

and
$$I = \int \frac{\tan^2\theta}{\sec^4\theta}\,.\,\sec^2\theta\,d\theta = \int \sin^2\theta\,d\theta$$

$$= \tfrac{1}{2}\int (1-\cos 2\theta)\,d\theta = \tfrac{1}{2}[\theta - \tfrac{1}{2}\sin 2\theta]$$

$$= \tfrac{1}{2}\left[\theta - \frac{\tan\theta}{1+\tan^2\theta}\right]$$

$$= \tfrac{1}{2}[\tan^{-1}x - x/(1+x^2)].$$

Thus the complete solution is

$$y(1+x^2)^{\frac{1}{2}} = \tfrac{1}{2}[\tan^{-1}x - x(1+x^2)^{-1}] + C,$$

i.e.
$$y = \tfrac{1}{2}(1+x^2)^{-\frac{1}{2}}[\tan^{-1}x - x(1+x^2)^{-1}] + C(1+x^2)^{-\frac{1}{2}}.$$

9. Equations reducible to linear equations.

By choosing z, a new variable, as some function of y, certain first-order differential equations can be reduced to linear equations of the form

$$\frac{dz}{dx} + Pz = Q,$$

where P and Q are functions of x, and then solved in the usual manner.

Example 13 (L.U.).—Solve the equation $dy/dx - 4y = xy^{3/2}$.

The equation can be written as

$$\frac{1}{y^{3/2}}\frac{dy}{dx} - 4\,.\,\frac{1}{y^{1/2}} = x. \quad \ldots\ldots\ldots \quad \text{(i)}$$

Let
$$z = \frac{1}{y^{1/2}}, \quad \therefore\ \frac{dz}{dx} = -\frac{1}{2y^{3/2}}\,.\,\frac{dy}{dx},$$

$$\therefore\ \frac{1}{y^{3/2}}\frac{dy}{dx} = -2\frac{dz}{dx},$$

and equation (i) becomes
$$-2\frac{dz}{dx} - 4z = x,$$

i.e.
$$\frac{dz}{dx} + 2z = -\frac{x}{2}. \quad \ldots\ldots\ldots \quad \text{(ii)}$$

The integrating factor of equation (ii) is $e^{\int 2dx} = e^{2x}$.

Multiplying equation (ii) by e^{2x},

$$e^{2x}\, dz/dx + 2ze^{2x} = -\tfrac{1}{2}xe^{2x},$$

$$\therefore\ d/dz(ze^{2x}) = -\tfrac{1}{2}xe^{2x}.$$

$$\begin{aligned}\therefore\ ze^{2x} &= -\tfrac{1}{2}\int xe^{2x}\, dx \\ &= -\tfrac{1}{2}[\tfrac{1}{2}xe^{2x} - \tfrac{1}{2}\int e^{2x}\, dx] + C \\ &\qquad\qquad \text{(integration by parts)} \\ &= \tfrac{1}{8}e^{2x} - \tfrac{1}{4}xe^{2x} + C.\end{aligned}$$

$$\therefore\ z = y^{-1/2} = \tfrac{1}{8} - \tfrac{1}{4}x + Ce^{-2x},$$

$$\therefore\ y = \frac{1}{(1/8 - x/4 + Ce^{-2x})^2}.$$

D. Special Cases

10. This covers all other types of first-order differential equation of the first degree that have not already been dealt with and may be met. The solution in these cases usually depends upon a substitution, which will be shown by means of examples.

Example 14.—Solve the equation $dy/dx = (2y + x - 1)^2$.

Let $$2y + x - 1 = z, \quad \therefore\ 2\frac{dy}{dx} + 1 = \frac{dz}{dx},$$

i.e. $$\frac{dy}{dx} = \tfrac{1}{2}\left(\frac{dz}{dx} - 1\right),$$

and the given equation becomes

$$\tfrac{1}{2}(dz/dx - 1) = z^2,$$

i.e. $$dz/dx = 1 + 2z^2.$$

$$\therefore\ \frac{1}{\tfrac{1}{2} + z^2}\frac{dz}{dx} = 2.$$

Integrating with respect to x,

$$1/(1/\sqrt{2}) \tan^{-1} z/(1/\sqrt{2}) = 2x + C,$$

i.e. $$\sqrt{2} \tan^{-1}\{\sqrt{2}\,.\,(2y + x - 1)\} = (2x + C).$$

Note.—All equations of the form $dy/dx = (ax + by + c)^n$, where $n = 2$, can be dealt with in this manner, and when $n = 3$, 4, etc., the solutions can be obtained by similar substitutions.

EXAMPLES ON CHAPTER XV

Solve the following differential equations under the conditions stated.

1. $\sqrt{(1+x)}\,dy/dx - 2\sqrt{(1-x)} = 0.$
2. $2\sqrt{(x)}\,dy/dx = x^2 - 1.$
3. $dy/dx + y = y^2.$
4. $dy/dx = 1 + y^2$, given $y = 1$ when $x = 0$.
5. $y\,dy/dx = \sqrt{(1+y^2)}.$
6. $x \cos y\,dy/dx - (1+x^2)\sin y = 0.$
7. $y(1+x^2)\,dy/dx - 2x(1-y^2) = 0.$
8. $xy\,dy/dx - (1+x)\sqrt{(y^2-1)} = 0.$
9. $2xy^2\,dy/dx - (1+y^2) = 0.$
10. $dy/dx + y/x = 1/x^2.$
11. $(x-2y-3)\,dy/dx = 2x - 4y - 2.$
12. $dy/dx + 4y = 64x^2 - 16$, where $y = 0$ when $x = 0$.
13. $\dfrac{dy}{dx} = \dfrac{x+y+1}{x+y-1}$, where $y = 2$ when $x = 0$.
14. $dy/dx = (y-x)^2$, where $y = 0$ when $x = 0$.
15. $dy/dx + y = x^2.$
16. $dy/dx + y \cot x = \sin 2x$, where $y = 0$ when $x = \pi/2$.
17. $(1+x)\,dy/dx + \beta xy = \beta e^{-\beta x}$, where $y = 0$ when $x = 0$.
18. $x\,dy/dx + 3y = 4x + 3.$
19. $y\,dy/dx = y^2 + 4x.$

The following questions are all taken from London University examination papers.

20. Solve the differential equations:

$$(a)\ \frac{dy}{dx} + \frac{xy}{1-x^2} = a \sin^{-1} x, \qquad (b)\ \frac{dy}{dx} + 2xy = x^3,$$

$$(c)\ \frac{dy}{dx} = \frac{2x+y-1}{x+2y+1}.$$

21. (i) Find the value of a, given $x^3\,dp/dx = a - x$ and $p = 0$ when $x = 2$ and $x = 6$.

(ii) Solve the equation $$\frac{dy}{dx} = \frac{x+y-3}{x+y+3},$$
given that $y = 1$ when $x = 0$.

22. Solve the differential equations:

$$\text{(i)}\ \frac{dy}{dx} - \frac{y}{1-x^2} = 1 + x, \qquad \text{(ii)}\ \frac{dy}{dx} = \frac{y(x+2y)}{x(2x+y)},$$

$$\text{(iii)}\ (x^2+x)\,dy/dx + y = 2x.$$

23. Solve the differential equations:

$$\text{(i) } dy/dx + 2y\tan x = \sin x,$$
$$\text{(ii) } (5y + 7x)\,dy/dx + (8y + 10x) = 0.$$

24. Solve the equations:

$$\text{(i) } (1 - x^2)\,dy/dx + 2xy = x - x^3, \qquad \text{(ii) } (3x - y)\,dy/dx = 2x,$$
$$\text{(iii) } 2x\frac{dy}{dx} = \frac{\sin x}{y} - y.$$

25. Show how to solve the differential equation $dy/dx + Py = Qy^n$, wher P and Q are functions of x only.

Solve the equation $xy\,dy/dx - 3y^2 = 3x^3$.

26. Solve the differential equations:

$$(a)\ \frac{dy}{dx} = \frac{2x^2 + xy + y^2}{x^2 + xy + 2y^2}, \qquad (b)\ y - 2x\frac{dy}{dx} = x(x + 1)y^3,$$
$$(c)\ \frac{dy}{dx} + \frac{4x^2 + 8xy + 5y^2}{y^2} = 0.$$

27. Solve the following equations:

$$(a)\ 4(x - 2)^2\,dy/dx = (x + y - 1)^2, \qquad (b)\ (1 + x^2)\,dy/dx - xy = 1 - x,$$
$$(c)\ y(y^2 - 2x^2) + x(2y^2 - x^2)\,dy/dx = 0.$$

28. Solve the differential equations:

$$(a)\ dy/dx - xy = x^3y^2, \qquad (b)\ (3x + 2y - 4)\,dy/dx = 3y - 2x + 7,$$
$$(c)\ dy/dx + y\tan x = y^3\sec^6 x.$$

29. Solve the following:

$$(a)\ x\frac{dy}{dx} = 2y + x^{n+1}y - x^n y^{3/2}, \qquad (b)\ -\frac{dy}{dx} = \frac{2x + y}{x + 3y + 5},$$
$$(c)\ (4x + 2y + 1)\,dy/dx = 2x + y - 3.$$

30. Solve the differential equations:

$$(a)\ dy/dx + y\cot x = y^2\cos^2 x, \qquad (b)\ (x^2 + y^2)\,dy/dx = x^2 + xy,$$
$$(c)\ (x - y)\,dy/dx = 2x + y - 3.$$

31. Solve the following equations:

$$(a)\ 2(1 + x)\,dy/dx - (1 + 2x)y = x^2\sqrt{(1 + x)},$$
$$(b)\ x(1 - x^2)\,dy/dx + (2x^2 - 1)y = x^3y^3,$$
$$(c)\ x\,dy/dx - y = (x^2 + y^2)^{\frac{1}{2}}.$$

32. Solve the differential equations:

$$(a)\ \frac{dy}{dx} = \frac{x + y - 3}{3x - y - 1}, \qquad (b)\ x^2\frac{dy}{dx} + xy + y = 0,$$
$$(c)\ \{(x - 1)^2 + (y - 2)^2\}\,dy/dx + 2(x - 1)^2 + 2(x - 1)(y - 2) = 0.$$

33. Show how to solve the equation $dy/dx + Py + Qy^n = 0$, where P and Q are functions of x.

Solve the equation $\cos x\,dy/dx + y\sin x + 2y^3 = 0$.

34. Find the differential equation of the family of curves which are such that a tangent at a point P meets the axis of y at a point T, so that OT = OP, where O is the origin of rectangular co-ordinates. Integrate this equation to find the Cartesian equation of the family.

35. Show that, if two curves, whose equations are referred to polar co-ordinates, cut at right angles at any point, then the value of $dr/d\theta$ at that point obtained for the equation of one curve is equal to the value of $-r^2\, d\theta/dr$ at that point obtained for the equation of the other curve.

Find the equation of the orthogonal trajectories of the family of cardioids $r = \lambda(1 - \cos\theta)$, where λ is a parameter.

36. The normal at a point P of a curve meets the x-axis at G, and N, the foot of the ordinate of P, lies between G and the origin O. If OG = OP, find the differential equation of the system of curves for which this condition holds, and integrate it.

Find also the equation of the orthogonal trajectories of the system, and show that they are parabolas.

37. The feet of the perpendiculars from the point $(c, 0)$ to the tangents to a certain curve lie on the circle $x^2 + y^2 = a^2$. Obtain the differential equation of the curve in the form $y^2 - 2xyp + (x^2 - a^2)p^2 + c^2 - a^2 = 0$, where $p = dy/dx$.

38. P is a point on a curve, and the tangent at P meets the x-axis in T; the normal at P meets the y-axis in M, and the line through P parallel to the x-axis meets the y-axis in L; L lies between the origin O, and M. If OT = LM, find the differential equation of the curve.

Integrate the equation, and find the equation of the curve passing through the point (1, 1).

39. Find the family of curves which have the property that the foot N of the ordinate of a point P on any member of the family bisects OG, where O is the origin and G is the intersection of the normal at P with the x-axis.

Find also the family of curves which have the property that NG = OP.

40. A family of curves possesses the following property: if P be any point on one of the curves, and the normal at P cuts the x-axis at G, then $PG^{\frac{1}{2}}$ varies as y, the ordinate of P. Find and solve the differential equation of the family, and find also the family of orthogonal curves.

41. The ordinate and normal from a point P of a curve meet the x-axis in the points N and G respectively, and O is the origin of co-ordinates. The rectangle contained by ON and NG is equal to the sum of the squares on OP and PN. Find the Cartesian equation of the curve if it is known that the curve passes through the point (1, 1).

CHAPTER XVI

Second-order and Partial Differential Equations

Second-order Differential Equations

1. There are various types of differential equation of the second order that will be encountered, and these are classified as follows, with A and B representing the constants of integration throughout.

N.B.—Second-order differential equations always give rise to a complete solution involving two arbitrary constants.

2. Type 1.—These are of the form $d^2y/dx^2 = f(x)$ and are solved by straightforward integration, giving

$$\frac{dy}{dx} = \int f(x)\,dx + A$$

and
$$y = \int\left\{\int f(x)\,dx\right\}dx + Ax + B.$$

3. Type 2.—These are of the form

$$\frac{d^2y}{dx^2} = f(y). \qquad (1)$$

The general method of solution of this type is to use

$$p = \frac{dy}{dx} \quad \left(\text{use } v \text{ for } \frac{dx}{dt} \text{ or } \frac{ds}{dt} \text{ giving velocity}\right),$$

then
$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\,\frac{dp}{dy}.$$

Thus equation (1) becomes $\quad p\,\dfrac{dp}{dy} = f(y). \qquad (2)$

Integrating (2) with respect to y,

$$\tfrac{1}{2}p^2 = \int f(y)\,dy + \tfrac{1}{2}A,$$

i.e. $$p^2 = F(y) + A \qquad \left[\text{where } F(y) = 2\int f(y)\,dy\right],$$

i.e. $$p = dy/dx = \pm\sqrt{\{F(y) + A\}},$$

$$\therefore \quad \frac{1}{\sqrt{\{F(y) + A\}}}\frac{dy}{dx} = \pm 1.$$

Integrating with respect to x,

$$\int \frac{dy}{\sqrt{\{F(y) + A\}}} = \pm (x + B).$$

which gives the solution after integration.

N.B.—The ambiguity in sign can generally be removed, when no boundary conditions are given, by squaring, or other methods dependent upon the particular question.

Example 1.—Solve the equation $\dfrac{d^2y}{dx^2} = -\dfrac{4}{y^3}$, given that $\dfrac{dy}{dx} = \tfrac{1}{4}$ and $y = 8$ when $x = 0$.

Let $$p = \frac{dy}{dx}, \quad \text{then} \quad \frac{d^2y}{dx^2} = p\frac{dp}{dy},$$

and the given equation becomes

$$p\frac{dp}{dy} = -\frac{4}{y^3}.$$

Integrating with respect to y,

$$\frac{p^2}{2} = \frac{2}{y^2} + A.$$

But $p = \tfrac{1}{4}$ when $y = 8$, $$\therefore \tfrac{1}{32} = \tfrac{1}{32} + A.$$

$$\therefore A = 0$$

and $$p^2 = 4/y^2,$$

i.e. $$p = \frac{dy}{dx} = +\frac{2}{y}, \quad \left[\begin{matrix}\text{positive sign since } p = \tfrac{1}{4} \\ \text{when } y = 8\end{matrix}\right],$$

i.e. $$y\frac{dy}{dx} = +2.$$

Integrating with respect to x,

$$\tfrac{1}{2}y^2 = 2x + B.$$

Now $y = 8$ when $x = 0$, $$\therefore B = 32,$$

$$\therefore y^2 = 4(x + 16).$$

Example 2.—Solve the equation $\dfrac{d^2x}{dt^2} = -\dfrac{2}{x^2}$, given $\dfrac{dx}{dt} = 0$ and $x = 4$ when $t = 0$.

Note.—This is an example of the inverse square law which frequently occurs in natural phenomena.

Let $$\frac{dx}{dt} = v, \quad \therefore \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx},$$

and the original equation becomes

$$v\frac{dv}{dx} = -\frac{2}{x^2}.$$

Integrating with respect to x,

$$\tfrac{1}{2}v^2 = 2/x + A.$$

But $v = dx/dt = 0$ when $x = 4$,

$$\therefore 0 = A + \tfrac{1}{2}, \quad \therefore A = -\tfrac{1}{2}.$$

Hence $$\tfrac{1}{2}v^2 = 2/x - \tfrac{1}{2},$$

i.e. $$v^2 = 4/x - 1.$$

$$\therefore \frac{dx}{dt} = v = \pm\sqrt{\left(\frac{4-x}{x}\right)},$$

i.e. $$\sqrt{\left(\frac{x}{4-x}\right)}\frac{dx}{dt} = \pm 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Let $$I = \int\sqrt{\left(\frac{x}{4-x}\right)}dx$$

and $\sqrt{x} = 2\sin\theta, \quad \therefore x = 4\sin^2\theta, \quad \therefore dx = 8\sin\theta\cos\theta\,d\theta.$

Hence $$I = \int\frac{2\sin\theta \,.\, 8\sin\theta\cos\theta\,d\theta}{2\cos\theta} = \int 8\sin^2\theta\,d\theta$$

$$= 4\int(1-\cos 2\theta)\,d\theta = 4[\theta - \tfrac{1}{2}\sin 2\theta]$$
$$= 4[\theta - \sin\theta\cos\theta] = 4[\theta - \sin\theta\sqrt{(1-\sin^2\theta)}]$$
$$= 4[\sin^{-1}\tfrac{1}{2}\sqrt{x} - \tfrac{1}{2}\sqrt{x}\,\sqrt{(1-x/4)}] = 4\sin^{-1}\tfrac{1}{2}\sqrt{x} - \sqrt{(4x-x^2)}.$$

Integrating (i) with respect to x,

$$I = \pm(t + B),$$

i.e. $$4\sin^{-1}\tfrac{1}{2}\sqrt{x} - \sqrt{(4x-x^2)} = \pm(t+B).$$

Now $x = 4$ when $t = 0$,

$$\therefore 4\sin^{-1}1 = \pm B, \quad \therefore B = \pm 2\pi.$$

Thus the solution required is

$$4\sin^{-1}\tfrac{1}{2}\sqrt{x} - \sqrt{(4x-x^2)} = \pm t + 2\pi.$$

Example 3.—(Simple harmonic motion.)

Solve the equation $\dfrac{d^2x}{dt^2} = -n^2x$.

Let $$\frac{dx}{dt} = v,$$

therefore, as previously, $$\frac{d^2x}{dt^2} = v\frac{dv}{dx}.$$

The equation of motion now becomes

$$v\frac{dv}{dx} = -n^2x.$$

Integrating with respect to x,

$$\tfrac{1}{2}v^2 = -\tfrac{1}{2}n^2x^2 + \tfrac{1}{2}n^2a^2$$

(where a is an arbitrary constant).

($N.B.$—Use n^2a^2, i.e. a positive constant, since v^2 is always positive).

Hence $$v^2 = n^2(a^2 - x^2),$$

$$\therefore \qquad \frac{1}{\sqrt{(a^2 - x^2)}}\frac{dx}{dt} = \pm n. \qquad \text{(i)}$$

[Negative sign used to give a simpler solution.]

Integrating equation (i) with respect to t,

$$\cos^{-1}(x/a) = \pm(nt + \varepsilon),$$

where ε is an arbitrary constant.

$$\therefore\ x = a\cos\pm(nt + \varepsilon),$$

i.e. $$x = a\cos(nt + \varepsilon).$$

N.B.—If the positive root were used in equation (i), an inverse sine would arise and the ambiguity in sign would not be resolved.

4. Type 3.—These are equations of the type

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0, \qquad (3)$$

where a and b are constants.

Consider $y = Ae^{kx}$ as a solution of equation (3), where A is an arbitrary constant.

Then $$\frac{dy}{dx} = Ake^{kx} = ky,$$

$$\frac{d^2y}{dx^2} = Ak^2e^{kx} = k^2y.$$

Using these in (3), $$y(k^2 + ak + b) = 0,$$

i.e. $$k^2 + ak + b = 0 \qquad (y \neq 0). \quad . \quad (4)$$

The equation (4) is known as the *auxiliary equation* of equation (3), and if k_1 and k_2 are its roots, then $y = Ae^{k_1x}$ and $y = Be^{k_2x}$ will be solutions of equation (3), where A and B are arbitrary constants, not necessarily the same.

The complete solution of equation (3) must contain two arbitrary constants, and therefore it is most probable that the complete solution will be

$$y = Ae^{k_1x} + Be^{k_2x},$$

where k_1 and k_2 are the roots of the auxiliary equation (4) and therefore

$$\left.\begin{aligned} k_1^2 + ak_1 + b = 0 \\ k_2^2 + ak_2 + b = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This is verified as follows:

If $$y = Ae^{k_1x} + Be^{k_2x},$$

then $$\frac{dy}{dx} = k_1Ae^{k_1x} + k_2Be^{k_2x},$$

and $$\frac{d^2y}{dx^2} = k_1^2Ae^{k_1x} + k_2^2Be^{k_2x}.$$

$$\begin{aligned} \therefore \frac{d^2y}{dx^2} + a\frac{dy}{dx} + by &= (k_1^2Ae^{k_1x} + k_2^2Be^{k_2x}) + a(k_1Ae^{k_1x} + k_2Be^{k_2x}) \\ &\qquad + b(Ae^{k_1x} + Be^{k_2x}) \\ &= Ae^{k_1x}(k_1^2 + ak_1 + b) + Be^{k_2x}(k_2^2 + ak_2 + b) \\ &= 0 \qquad \text{[using results (5)].} \end{aligned}$$

$$\therefore y = Ae^{k_1x} + Be^{k_2x}$$

satisfies equation (3) and is a complete solution of it.

There are three different cases to be considered dependent upon the nature of the roots of the auxiliary equation $k^2 + ak + b = 0$.

Case (i).—Auxiliary equation with *real unequal* roots.

This is a straightforward case and will be illustrated by means of an example.

Hence the complete solution is

$$y = e^{\frac{1}{2}x}(A + Bx).$$

Now $y = 1$ when $x = 0$, $\therefore\ 1 = A$,

and $y = 2$ when $x = 1$, $\therefore\ 2 = e^{\frac{1}{2}}(A + B)$.

$$\therefore\ A + B = 2e^{-\frac{1}{2}},$$

i.e. $$B = 2e^{-\frac{1}{2}} - 1.$$

The required solution is

$$y = e^{\frac{1}{2}x}\{1 + (2e^{-\frac{1}{2}} - 1)x\},$$

i.e. $$y = e^{\frac{1}{2}x}(1 - x) + 2e^{\frac{1}{2}(x-1)}.$$

5. Type 4.—Homogeneous equations.

Equations of the type

$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = f(x),$$

where a and b are constants, are known as homogeneous equations of the second order, and there are two methods of solution.

Method I.—This is only suitable when $f(x) = 0$, and the equation becomes

$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0. \quad \ldots \ldots \quad (8)$$

Assume that $y = Ax^k$ is a solution of equation (8). Then

$$\frac{dy}{dx} = kAx^{k-1} \text{ and } \frac{d^2y}{dx^2} = k(k-1)Ax^{k-2}.$$

Using these in equation (8), it becomes

$$k(k-1)Ax^k + akAx^k + bAx^k = 0,$$

i.e. $$Ax^k\{k(k-1) + ak + b\} = 0,$$

i.e. $$k(k-1) + ak + b = 0 \quad \ldots \ldots \quad (9)$$

(if $Ax^k \neq 0$).

As in Type 3, the equation (9) is known as the *auxiliary equation* and, if k_1 and k_2 be the roots of this equation (9), the complete solution is

$$y = Ax^{k_1} + Bx^{k_2}.$$

This solution can be verified as in Type 3.

N.B.—To form the auxiliary equation, replace $x^2(d^2y/dx^2)$ by $k(k-1)$, $x(dy/dx)$ by k, and leave out the y.

Method II.—The case when $f(x) \neq 0$ will be dealt with later, and the case to be considered now is the equation

$$x^2 \frac{d^2y}{dx^2} + ax \frac{dy}{dx} + by = 0.$$

Let $$x = e^t, \quad \therefore \frac{dx}{dt} = e^t = x, \quad \therefore \frac{dt}{dx} = \frac{1}{x}.$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt},$$

$$\therefore x \frac{dy}{dx} = \frac{dy}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Differentiating equation (10) with respect to x,

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right) \cdot \frac{dt}{dx} = \frac{1}{x} \frac{d^2y}{dt^2}.$$

$$\therefore x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = \frac{d^2y}{dt^2}.$$

$$\therefore x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt} \text{ [using (10)]}. \quad . \quad (11)$$

Substituting from equations (10) and (11) in equation (8), it becomes

$$\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + a \frac{dy}{dt} + by = 0. \quad . \quad . \quad . \quad . \quad (12)$$

If $y = Ae^{kt}$ be a solution of equation (12), which is of Type 3, the auxiliary equation is

$$k^2 - k + ak + b = 0,$$

i.e. $$k(k - 1) + ak + b = 0,$$

which is the same as the auxiliary equation in Method I.

If k_1 and k_2 be the roots of this auxiliary equation, the complete solution is

$$y = Ae^{k_1t} + Be^{k_2t},$$

i.e. $$y = Ax^{k_1} + Bx^{k_2} \qquad \text{(as previously)}.$$

N.B.—This method of attack is the only one possible when $f(x) \neq 0$.

Therefore the solution is given by

$$\left.\begin{aligned} x &= Ae^{\frac{1}{3}t} + Be^{-\frac{1}{3}t}, \\ y &= \tfrac{1}{2}e^{t} - 2Ae^{\frac{1}{3}t} - Be^{-\frac{1}{3}t}. \end{aligned}\right\}$$

Example 28.—Solve the simultaneous differential equations:

$$\frac{dx}{dt} + 4x + 3y = t,$$

$$\frac{dy}{dt} + 2x + 5y = e^{t}.$$

Using the operator D for d/dt, the equations become

$$(D + 4)x + 3y = t \quad \ldots \ldots \ldots \quad \text{(i)}$$

$$2x + (D + 5)y = e^{t}. \quad \ldots \ldots \ldots \quad \text{(ii)}$$

$(D + 5)$ operating on (i) gives

$$(D + 5)(D + 4)x + 3(D + 5)y = 5t + 1. \quad \ldots \ldots \quad \text{(iii)}$$

(ii) $\times$ 3 gives

$$6x + 3(D + 5)y = 3e^{t}. \quad \ldots \ldots \ldots \quad \text{(iv)}$$

From (iii) $-$ (iv),

$$(D^2 + 9D + 14)x = 5t + 1 - 3e^{t}. \quad \ldots \ldots \quad \text{(v)}$$

Let u be the complementary function and v the particular integral for equation (v), then

$$(D^2 + 9D + 14)u = 0, \quad \ldots \ldots \ldots \quad \text{(vi)}$$

and

$$v = \frac{1}{D^2 + 9D + 14}(5t + 1 - 3e^{t}). \quad \ldots \ldots \quad \text{(vii)}$$

From (vi), using $u = Ae^{kt}$, the auxiliary equation is

$$k^2 + 9k + 14 = 0,$$

from which

$$k = -7 \text{ or } -2.$$

$$\therefore\ u = Ae^{-7t} + Be^{-2t}.$$

From (vii),

$$\begin{aligned} v &= \frac{1}{D^2 + 9D + 14}(5t + 1) - 3\,\frac{1}{D^2 + 9D + 14}\,e^{t} \\ &= \frac{1}{14\{1 + (9D + D^2)/14\}}(5t + 1) - \frac{3}{24}\,e^{t} \\ &= \tfrac{1}{14}\left\{1 - \frac{9D + D^2}{14} + \ldots\right\}(5t + 1) - \tfrac{1}{8}e^{t} \\ &= \tfrac{1}{14}(5t - \tfrac{45}{14} + 1) - \tfrac{1}{8}e^{t} \\ &= \tfrac{5}{14}t - \tfrac{31}{196} - \tfrac{1}{8}e^{t}. \end{aligned}$$

$$\therefore\ x = Ae^{-7t} + Be^{-2t} + \tfrac{5}{14}t - \tfrac{31}{196} - \tfrac{1}{8}e^{t}.$$

From (i),

$$\begin{aligned} 3y &= t - \left\{\begin{aligned} &-7Ae^{-7t} - 2Be^{-2t} - \tfrac{1}{8}e^{t} + \tfrac{5}{14} \\ &\quad + 4Ae^{-7t} + 4Be^{-2t} - \tfrac{1}{2}e^{t} + \tfrac{10}{7}t - \tfrac{31}{49} \end{aligned}\right\} \\ &= 3Ae^{-7t} - 2Be^{-2t} + \tfrac{5}{8}e^{t} + \tfrac{27}{98} - \tfrac{3}{7}t. \end{aligned}$$

$$\therefore\ y = Ae^{-7t} - \tfrac{2}{3}Be^{-2t} + \tfrac{5}{24}e^{t} - \tfrac{1}{7}t + \tfrac{9}{98}.$$

Example 29 (L.U.).—In a heat exchange, the temperatures θ and τ of the two liquids satisfy the equations

$$m\frac{d\theta}{dx} = k(\tau - \theta) = \frac{d\tau}{dx},$$

where m and k are constants. Find the general solution.

If $m = 2$, $k = 0{\cdot}5$, $\theta = 20$ when $x = 0$, and $\tau = 100$ when $x = 3$, find θ when $x = 3$, and τ when $x = 0$.

The given equations can be written as

$$(mD + k)\theta - k\tau = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$(D - k)\tau + k\theta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

$(D - k)$ operating on (i) gives

$$(mD^2 + kD - mkD - k^2)\theta - k(D - k)\tau = 0. \quad . \quad . \quad . \quad \text{(iii)}$$

(ii) $\times$ k gives $$k^2\theta + k(D - k)\tau = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

(iii) + (iv) gives $$\{mD^2 + (k - mk)D\}\theta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

If $\theta = Ae^{pt}$ be a solution of equation (v), the auxiliary equation is

$$mp^2 + (k - mk)p = 0.$$

$$\therefore\ p = 0 \text{ or } k(m - 1)/m.$$

$$\therefore\ \theta = A + Be^{k(m-1)x/m}.$$

From equation (i), using this value of θ,

$$\tau = (1/k)(mD + k)(A + Be^{k(m-1)x/m})$$

$$= (1/k)\{Bk(m - 1)e^{k(m-1)x/m} + kA + Bke^{k(m-1)x/m}\},$$

i.e. $$\tau = mBe^{k(m-1)x/m} + A.$$

Now $m = 2$, $k = 0{\cdot}5$; $\theta = 20$ when $x = 0$; $\tau = 100$ when $x = 3$,

$$\therefore\ \theta = A + Be^{\frac{1}{4}x},$$

and $$\tau = A + 2Be^{\frac{1}{4}x},$$

also $$20 = A + B, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

and $$100 = A + 2Be^{\frac{3}{4}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vii)}$$

From (vi) and (vii), $$B = \frac{80}{2e^{\frac{3}{4}} - 1},$$

$$A = 20 - \frac{80}{2e^{\frac{3}{4}} - 1} = \frac{40e^{\frac{3}{4}} - 100}{2e^{\frac{3}{4}} - 1}.$$

When $x = 3$, $$\theta = A + Be^{\frac{3}{4}} = \frac{-100 + 40e^{\frac{3}{4}} + 80e^{\frac{3}{4}}}{2e^{\frac{3}{4}} - 1}$$

$$= \frac{20(6e^{\frac{3}{4}} - 5)}{2e^{\frac{3}{4}} - 1}.$$

Using these in equation (ii), the required solution is

$$y = (1 - 2x^2) + x\sqrt{(1 - x^2)}.$$

Example 32 (L.U.).—Transform the equation

$$(1 + x^2)^3 \frac{d^2y}{dx^2} + 2x(1 + x^2)^2 \frac{dy}{dx} + (1 + x^2)y = 3x$$

by the substitution $\qquad x = \tan\theta.$

Hence, or otherwise, determine the solution of this equation for which both y and dy/dx vanish when $x = 0$.

$$x = \tan\theta, \quad \therefore \frac{dx}{d\theta} = \sec^2\theta = 1 + x^2.$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{1}{1 + x^2} \cdot \frac{dy}{d\theta},$$

$$\therefore (1 + x^2)\frac{dy}{dx} = \frac{dy}{d\theta}.$$

Differentiating this with respect to x,

$$(1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{d^2y}{d\theta^2}\frac{d\theta}{dx}$$

$$= \frac{1}{1 + x^2}\frac{d^2y}{d\theta^2}.$$

$$\therefore (1 + x^2)^3 \frac{d^2y}{dx^2} + 2x(1 + x^2)^2\frac{dy}{dx} = (1 + x^2)\frac{d^2y}{d\theta^2}.$$

Using this result, the given equation becomes

$$(1 + x^2)\frac{d^2y}{d\theta^2} + (1 + x^2)y = 3x,$$

i.e.

$$\frac{d^2y}{d\theta^2} + y = \frac{3x}{1 + x^2} = \frac{3\tan\theta}{\sec^2\theta}$$

$$= 3\sin\theta\cos\theta = \tfrac{3}{2}\sin 2\theta. \qquad \text{(i)}$$

If u be the complementary function, and v the particular integral of equation (i),

$$\frac{d^2u}{d\theta^2} + u = 0, \qquad \text{(ii)}$$

and

$$v = \frac{1}{D^2 + 1} \cdot \tfrac{3}{2}\sin 2\theta. \qquad \text{(iii)}$$

From equation (ii) it is readily seen that

$$u = A\cos\theta + B\sin\theta.$$

From (iii), $$v = \frac{1}{-4+1} \cdot \tfrac{3}{2} \sin 2\theta = -\tfrac{1}{2} \sin 2\theta.$$

Thus the complete solution is $y = u + v$,

i.e. $$y = A \cos\theta + B \sin\theta - \tfrac{1}{2} \sin 2\theta.$$

Using trigonometry,

$$y = A \cdot \frac{1}{\sqrt{(1+x^2)}} + B \cdot \frac{x}{\sqrt{(1+x^2)}} - \tfrac{1}{2} \cdot \frac{2x}{1+x^2}$$

$$= \frac{A}{\sqrt{(1+x^2)}} + \frac{Bx}{\sqrt{(1+x^2)}} - \frac{x}{1+x^2}.$$

Now $y = 0$ when $x = 0$, $\therefore A = 0$,

and $$y = \frac{Bx}{\sqrt{(1+x^2)}} - \frac{x}{1+x^2}.$$

$$\therefore \frac{dy}{dx} = B\frac{\{\sqrt{(1+x^2)} - x^2/\sqrt{(1+x^2)}\}}{(1+x^2)} - \frac{(1+x^2) - 2x^2}{(1+x^2)^2}.$$

But $\dfrac{dy}{dx} = 0$ when $x = 0$, $\therefore 0 = B - 1$,

$\therefore B = 1.$

Hence required solution is

$$y = \frac{x}{1+x^2}\{\sqrt{(1+x^2)} - 1\}.$$

Partial Differential Equations

16. These are differential equations involving partial differential coefficients; they will not be dealt with in detail, and a few examples will suffice to show the methods to be applied in the problems likely to be encountered.

Example 33 (L.U.).—Show that the equation $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = 0$ may be transformed by the substitution $x = u + v$, $y = u - v$, into the equation $\dfrac{\partial^2 z}{\partial u \partial v} = 0$.

Solve the equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \sin \tfrac{1}{2}(x+y) - \sin \tfrac{1}{2}(x-y).$$

Since $x = u + v$, and $y = u - v$,

$$u = \tfrac{1}{2}(x+y), \qquad \text{(i)}$$

$$v = \tfrac{1}{2}(x-y). \qquad \text{(ii)}$$

Using equations (i) and (ii),

$$\frac{\partial u}{\partial x} = \tfrac{1}{2},\ \frac{\partial u}{\partial y} = \tfrac{1}{2},\ \frac{\partial v}{\partial x} = \tfrac{1}{2},\ \frac{\partial v}{\partial y} = -\tfrac{1}{2}. \quad \ldots \ldots \quad \text{(iii)}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \tfrac{1}{2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \qquad \text{[using (iii)].}$$

$$\therefore\ \frac{\partial^2 z}{\partial x^2} = \tfrac{1}{2}\left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\right\}$$

$$= \tfrac{1}{2}\left\{\left(\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial v}{\partial x}\right) + \left(\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}\right)\right\}$$

$$= \tfrac{1}{4}\left\{\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2}\right\} \quad \ldots \ldots \quad \text{(iv)}$$

[using (iii)].

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \tfrac{1}{2}\left\{\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right\} \qquad \text{[using (iii)].}$$

$$\therefore\ \frac{\partial^2 z}{\partial y^2} = \tfrac{1}{2}\left\{\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)\right\}$$

$$= \tfrac{1}{2}\left\{\left(\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial v}{\partial y}\right) - \left(\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial y}\right)\right\}$$

$$= \tfrac{1}{4}\left\{\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2}\right\} \quad \ldots \ldots \quad \text{(v)}$$

[using (iii)].

From (iv) — (v),
$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u\partial v}.$$

Thus the given equation transforms to

$$\frac{\partial^2 z}{\partial u\partial v} = 0.$$

Using this transformation, the equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \sin\tfrac{1}{2}(x+y) - \sin\tfrac{1}{2}(x-y)$$

becomes
$$\frac{\partial^2 z}{\partial u\partial v} = \sin u - \sin v.$$

Integrating partially with respect to u (i.e. considering v as a constant),

$$\frac{\partial z}{\partial v} = -\cos u - u\sin v + f(v),$$

where $f(v)$ is an arbitrary function of v.

Integrating this result partially with respect to v (i.e. considering u as a constant),

$$z = -v\cos u + u\cos v + f_1(v) + f_2(u),$$

where $f_1(v) = \int f(v)\,dv =$ an arbitrary function of v, and $f_2(u)$ is an arbitrary function of u.

Hence the solution is

$$z = u\cos v - v\cos u + f_1(v) + f_2(u),$$

i.e. $z = \frac{1}{2}(x+y)\cos\frac{1}{2}(x-y) - \frac{1}{2}(x-y)\cos\frac{1}{2}(x+y) + f_1(x-y) + f_2(x+y)$.

Example 34 (L.U.).—If $V = X\cos ay$, where a is a constant and X is a function of x only, satisfy the equation $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$, find the general form of the function X.

Find a *particular* solution of the equation that tends to zero as x tends to infinity, and is equal to $2\cos y + \cos 2y$ when $x = 0$.

$$V = X\cos ay, \quad \therefore \frac{\partial V}{\partial x} = \frac{dX}{dx}\,.\cos ay \quad\text{and}\quad \frac{\partial^2 V}{\partial x^2} = \frac{d^2X}{dx^2}\,.\cos ay.$$

Also $$\frac{\partial V}{\partial y} = -aX\sin ay \quad\text{and}\quad \frac{\partial^2 V}{\partial y^2} = -a^2X\cos ay.$$

$$\therefore \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \dots\dots\dots \quad \text{(i)}$$

becomes $$\frac{d^2X}{dx^2}\cos ay - a^2X\cos ay = 0,$$

i.e. $$\frac{d^2X}{dx^2} - a^2X = 0 \qquad (\cos ay \neq 0). \quad . \quad . \quad \text{(ii)}$$

The solution of equation (ii) is readily seen to be

$$X = Ae^{ax} + Be^{-ax}.$$

A solution of equation (i) is

$$V = (Ae^{ax} + Be^{-ax})\cos ay.$$

If $V \to 0$ as $x \to \infty$, then $A = 0,$

i.e. $V = Be^{-ax}\cos ay.$

To obtain $2\cos y + \cos 2y$ for V, when $x = 0$, requires two different cosines in the value of V.

Clearly, $V = B_1e^{-ax}\cos ay + B_2e^{-bx}\cos by$

will be a solution of equation (i), giving $V \to 0$ as $x \to \infty$, where B_1 and B_2 are arbitrary constants.

But $V = 2\cos y + \cos 2y$ when $x = 0$,

$$\therefore\ 2\cos y + \cos 2y \equiv B_1\cos ay + B_2\cos by.$$

This will be satisfied if $B_1 = 2$, $a = 1$, and $B_2 = 1$, $b = 2$.
Hence the required particular solution will be

$$V = 2e^{-x}\cos y + e^{-2x}\cos 2y.$$

Example 35.—Transform the equation $\dfrac{\partial^2 y}{\partial t^2} = a^2\dfrac{\partial^2 y}{\partial x^2}$ by changing the independent variables x and t to u and v by means of the relationships $u = x - at$, $v = x + at$. Hence solve the original equation.

$$u = x - at, \quad v = x + at.$$

$$\therefore \frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial t} = -a, \quad \frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial t} = a. \quad \ldots \quad \text{(i)}$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial t}$$

$$= a\left\{\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\right\} \qquad \text{[using (i)].}$$

$$\frac{\partial^2 y}{\partial t^2} = a\left\{\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial v}\right) - \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial u}\right)\right\}$$

$$= a\left\{\left(\frac{\partial^2 y}{\partial v^2}\cdot\frac{\partial v}{\partial t} + \frac{\partial^2 y}{\partial u \partial v}\cdot\frac{\partial u}{\partial t}\right) - \left(\frac{\partial^2 y}{\partial v \partial u}\cdot\frac{\partial v}{\partial t} + \frac{\partial^2 y}{\partial u^2}\cdot\frac{\partial u}{\partial t}\right)\right\}$$

$$= a^2\left\{\frac{\partial^2 y}{\partial v^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial u^2}\right\} \quad \ldots \quad \text{(ii)} \quad \text{[using (i)].}$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \qquad \text{[using (i)].}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial v}\right)$$

$$= \left(\frac{\partial^2 y}{\partial u^2}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 y}{\partial v \partial u}\cdot\frac{\partial v}{\partial x}\right) + \left(\frac{\partial^2 y}{\partial u \partial v}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 y}{\partial v^2}\cdot\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \quad \ldots \quad \text{(iii)} \quad \text{[using (i)].}$$

With the values in (ii) and (iii), the given equation becomes

$$a^2\left\{\frac{\partial^2 y}{\partial v^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial u^2}\right\} = a^2\left\{\frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right\},$$

which reduces to
$$\frac{\partial^2 y}{\partial u \partial v} = 0 \qquad (\text{since } a \neq 0).$$

The solution of this is obtained by integrating partially with respect to u (keeping v constant), giving

$$\frac{\partial y}{\partial v} = f(v),$$

where $f(v)$ is an arbitrary function; now integrating the result partially with respect to v gives

$$y = \int f(v)\,dv + \phi(u) = F(v) + \phi(u),$$

where $F(v)$ and $\phi(u)$ are arbitrary functions. Therefore the required solution is

$$y = F(x + at) + \phi(x - at).$$

EXAMPLES ON CHAPTER XVI

All the following examples are taken from London University examination papers.

1. Solve (i) $\dfrac{d^2y}{dx^2} + y = 3 - 2x^2$, given that $y = 7$ and $dy/dx = 0$ when $x = 0$;

(ii) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 10y = \sin x.$

2. Solve the equations

$$\text{(i) } \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{-x}\sin 2x,$$

$$\text{(ii) } \frac{d^2y}{dx^2} = 2(y^3 + y),$$

under the conditions that $y = 0$ and $dy/dx = 1$ when $x = 0$.

3. Solve the equations

$$\text{(i) } \frac{d^2y}{dx^2} + 4y = \cos x \cos 3x,$$

$$\text{(ii) } \frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 10e^{-4x}.$$

4. Solve the equations (i) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = x^2,$

$$\text{(ii) } \frac{d^2y}{dx^2} - y = \sin 2x \sin x.$$

5. Solve the following equations subject to the conditions that when $x = 0$, $y = 2$ and $dy/dx = -1$,

$$(a)\ \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \sin x,$$

$$(b)\ \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x.$$

6. Solve the following differential equations,

$$\text{(i) } 4\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 3y = e^{-x}\sin \tfrac{1}{2}x,$$

$$\text{(ii) } \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = \sin 2x.$$

7. (i) Solve the equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^{-x}(1 + \sin x).$$

(ii) By means of the substitution $x = e^t$, or otherwise, solve the equation

$$9x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = x\log_e x.$$

8. (i) Solve the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = e^{2x}\sin 3x.$$

(ii) By means of the substitution $x = e^t$, or otherwise, solve the equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^3.$$

9. (i) Solve the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}(1 + x).$$

(ii) By means of the substitution $x = e^t$, or otherwise, solve the equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x.$$

10. (i) Solve the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = e^{2x}\sin x.$$

(ii) By means of the transformation $x = e^t$ (or otherwise), solve the equation

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x^2\log_e x.$$

11. (i) Solve the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^x(x + \sin x).$$

(ii) By the substitution $x = e^t$, or otherwise, solve the differential equation

$$4x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x + \log x.$$

12. Solve the equations

$$\text{(i)}\ \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = e^x\cos 2x,$$

$$\text{(ii)}\ \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = \sin 2x.$$

13. Show that the solution of the equation

$$\frac{d^2y}{dt^2} + 2n\frac{dy}{dt} + n^2y = A\cos pt,$$

for which both y and dy/dt vanish when $t = 0$, can be written

$$y = A\{\cos(pt - \phi) - e^{-nt}(nt + \cos\phi)\}/(n^2 + p^2),$$

where $$\tan\phi = 2np/(n^2 - p^2).$$

14. Solve the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 10\sin t,$$

subject to the conditions that y and dy/dt are both zero when t is zero.

15. Obtain the solution of any two of the following differential equations.

(i) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 1 + x + x^2;$

(ii) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = \sin 2x$, where $y = 0$ and $dy/dx = 0$ when $x = 0$;

(iii) $2\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 4y$, where $y = 1$ and $dy/dx = 0$ when $x = 0$.

$$\left[\textit{Hint.}\text{—Use } \frac{dy}{dx} = p, \text{ then } 2\frac{d^2y}{dx^2} = \frac{d}{dy}(p^2).\right]$$

16. Obtain the general solutions of the equations

$$\text{(i)}\quad \frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sin 2x,$$

$$\text{(ii)}\quad \frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + a^2y = x^2e^{-ax}.$$

17. (i) Solve the equation $\dfrac{d}{dt}\left(x\dfrac{dx}{dt}\right) = 6x,$

subject to the conditions that x and dx/dt both vanish when $t = 0$.

(ii) Solve the equation

$$\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + \alpha^2y = 12x^2e^{-\alpha x},$$

with the conditions that $y = 1$ and $dy/dx = 0$ when $x = 0$.

18. Find a quadratic polynomial which is a solution of the differential equation

$$x\frac{d^2y}{dx^2} - (x + 4)\frac{dy}{dx} + 2y = 0.$$

Find another solution in the form of a series

$$\sum_{n=0}^{\infty} a_n x^{n+5}, \text{ where } \frac{a_{n+1}}{a_n} = \frac{n+3}{(n+1)(n+6)}.$$

19. Show that the differential equation

$$x(1 + x)\frac{d^2y}{dx^2} + 3(1 + 2x)\frac{dy}{dx} + 6y = 0$$

has a solution A/x^2.

Assuming there is another solution in the form of an infinite series $\sum_0^{\infty} a_r x^r$, find a_r in terms of a_0, and hence sum the series assuming it is convergent.

20. Show that the differential equation

$$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = 0$$

admits of a solution which is a polynomial of the third degree in x. Show also that there is another solution in the form of an infinite series

$$a_0 + a_2x^2 + \ldots + a_{2r}x^{2r} + \ldots,$$

and find the relation between a_{2r+2} and a_{2r}.

21. Find in the form of a series, proceeding by ascending powers of x, one solution of the equation

$$x\frac{d^2y}{dx^2} + \frac{3}{2}\frac{dy}{dx} + y = 0.$$

22. Transform the equation

$$\frac{d^2y}{dx^2}\cos x + \frac{dy}{dx}\sin x - 2y\cos^3 x = 2\cos^5 x$$

into one having z as independent variable, where $z = \sin x$, and solve the equation.

23. Solve the differential equation

$$(1 + x^2)^2\frac{d^2y}{dx^2} + 2x(1 + x^2)\frac{dy}{dx} + 4y = 0$$

by changing the independent variable. [*Hint.*—Use $x = \tan\theta$.]

24. Solve the simultaneous differential equations

$$\frac{dx}{dt} + x - y = te^t,$$

$$2y - \frac{dx}{dt} + \frac{dy}{dt} = e^t.$$

25. Solve the equations

$$\frac{dx}{dt} + y = \sin t,$$

$$\frac{dy}{dt} + x = \cos t,$$

subject to the conditions that $x = 2$ and $y = 0$ when $t = 0$.

26. Solve the simultaneous differential equations

$$\frac{dx}{dt} + 2x + y = 0,$$

$$\frac{dy}{dt} + x + 2y = 0,$$

subject to the conditions that $x = 1$ and $y = 0$ when $t = 0$.

27. The currents i_1 and i_2 in the primary and secondary windings of a transformer are given by

$$L\frac{di_1}{dt} + M\frac{di_2}{dt} - Ri_1 = Ee^{ipt},$$

$$M\frac{di_1}{dt} + N\frac{di_2}{dt} - Si_2 = 0,$$

where L, M, N, R, S, E, p are given constants, and t is the time.

If, for sufficiently large values of t, it can be assumed that $i_1 = Ae^{ipt}$ and $i_2 = Be^{ipt}$, where A and B are complex constants, show that

$$\left|\frac{B}{A}\right| = \frac{Mp}{\sqrt{(S^2 + N^2p^2)}}.$$

Find A and B when $R = 0$ and $2LN = M^2$.

28. By using $z = u + iv$, or otherwise, solve the simultaneous equations

$$m\frac{du}{dt} = eE - evH,$$

$$m\frac{dv}{dt} = euH,$$

where m, e, E, H are constants.

If $u = dx/dt$ and $v = dy/dt$, find x and y as a function of the time t from your solution for z, and show further that if $x = y = u = v = 0$, when $t = 0$,

$$x = \frac{E}{\omega H}(1 - \cos \omega t), \quad y = \frac{E}{\omega H}(\omega t - \sin \omega t),$$

where $\omega = eH/m$.

29. The displacements of two connected masses at time t are given by the equations

$$3\frac{dx}{dt} + \frac{dy}{dt} + 2x = 3 \cos t,$$

$$2\frac{dy}{dt} + \frac{dx}{dt} + 3y = 7 \cos t - 4 \sin t.$$

Integrate these equations and determine the constants, given that x and y are both zero when $t = 0$.

30. Solve the simultaneous differential equations

$$\frac{dx}{dt} + 5x - 2y = t,$$

$$\frac{dy}{dt} + 2x + y = 0,$$

being given that $x = 0$ and $y = 0$ when $t = 0$.

31. Solve the simultaneous differential equations

$$\frac{dx}{dt} + 3x - 2y = 1,$$

$$\frac{dy}{dt} - 2x + 3y = e^t,$$

given that when $t = 0$, $x = y = 0$.

32. Obtain the most general solution of the partial differential equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

Obtain a solution of this equation satisfying the conditions $z = e^x - 1$ and $\partial z/\partial t = e^x$ when $t = 0$ [see Example 35, p. 498].

33. Find the first-order partial differential equation that is satisfied by $y^2 - xz = f(x^2 - yz)$, where z is the dependent variable and f the arbitrary function.

If u and v be functions of x, y, and z, and $u = f(v)$, where f represents the

arbitrary function, form the partial differential equation of the first order by eliminating the arbitrary function, taking x and y as the independent variables, and z as the dependent variable.

34. Obtain a solution of the partial differential equation

$$\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y,$$

for which $\frac{\partial z}{dy} = -2 \sin y$ when $x = 0$, and $z = 0$ when y is an odd multiple of $\pi/2$.

Solve the equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \sin (x + y) \cos (x - y)$$

[see Example 33].

35. Find a form for Y, a function of y only, so that $Y \cos px$ may be a solution of the partial differential equation

$$\frac{\partial z}{\partial y} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

36. If $x = ce^{-z} \sin \theta$ and $y = ce^{-z} \cos \theta$, change the variables from (x, y) to (z, θ) in the equation

$$y^2 \frac{\partial^2 u}{\partial x^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + x^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence show that the equation has a solution in the form

$$u = \Sigma A_n \cos (n\theta + \alpha) e^{-n^2 z},$$

where A, α, n are constants.

37. If $u = \frac{1}{r} f(ct - r) + \frac{1}{r} F(ct + r)$, where f and F denote arbitrary functions, show that

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} \right).$$

If u be also of the form given by $\frac{\cos r}{r} \phi(t)$, find the form of $\phi(t)$.

38. Find the form of Y, a function of y only, if $Y \sin mx$ satisfies the differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

Find a solution of the equation that vanishes when $y = -1$ and is equal to $\sin x$ when $y = 1$.

39. Transform the partial differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

to the form

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} = 0,$$

where $x = r \cos \theta$, $y = r \sin \theta$.

(i) If V be a function of r only, find the most general form of V satisfying the equation.

(ii) If $V = r^n f(\theta)$, where n is a constant, find the most general form of V satisfying the equation.

40. (i) If $r = \sqrt{(x^2 + y^2)}$ and $u = f(r)$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r),$$

and find u in terms of r if $\qquad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

(ii) If $x = r\cos\theta$, $y = r\sin\theta$, $u = r^\alpha \cos\theta$, and $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, find the possible values of the constant α.

41. If $V = f(z)$ and $z^2 = x^2/4t$, find $\dfrac{\partial V}{\partial t}$ and $\dfrac{\partial^2 V}{\partial x^2}$.

Show that if V is to satisfy the equation

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}, \text{ then } f(z) = A\int e^{-z^2}\, dz + B.$$

42. The current i in a circuit containing inductance, resistance, and capacitance is given by the equation

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} i = E\cos pt,$$

where L, R, C, E, and p are constants, and t is the time.

Show that, if R be positive, then the exponential terms in the solution of this equation tend to zero as t tends to infinity, and find the solution assuming these terms can be ignored.

With this assumption, show that for given values of E, p, and R the largest values of i occur when $\sqrt{(LC)} = 1/p$.

43. Find the particular integral of

$$\frac{d^2 x}{dt^2} + 2h\frac{dx}{dt} + (h^2 + p^2)x = ke^{-ht}\cos pt,$$

where x is distance, t is time, and h, p, k are constants, showing that it represents an oscillation of variable amplitude.

Show that this amplitude is a maximum when $ht = 1$.

44. (i) The equations of motion of electric charge in a circuit of self-inductance L, resistance R, and capacitance C are

$$L\frac{di}{dt} + Ri = -\frac{q}{C}, \quad \frac{dq}{dt} = i,$$

where i is the current and q the charge on the condenser. Find the condition that the discharge should be oscillatory.

(ii) Find the complete solution of the equation

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = E\cos pt,$$

where L, R, C, E, p are positive constants, and $CR^2 - 4L = -CN^2$, where N is a constant.

ANSWERS TO EXAMPLES

CHAPTER I (p. 8)

1. 19/4, 13/4.

2. (i) $+4$ to $3+\sqrt{3}$, and $3-\sqrt{3}$ to $+2$. (ii) $-5+2\sqrt{10}$.

4. $b^2 < 4ac$.

5. $-1 < x < 1$, $x < -2$, and $x > +2$. **6.** a +ve, $b^2 < 4ac$; 0 and -8.

7. $(x^2-2x+4)^2(x^2+4x+4)$; -2 twice, $(1 \pm 3i)$ twice.

8. (i) $16b^2(ab-1)/a^2$. **9.** (ii) $-7/6 < x < -1$, $x < -2$. **10.** $n3^{n+1}$.

11. $A = 16/5$, $B = -8/3$, $C = 7/15$; $n(2n-1)(2n+1)(12n^2-7)/15$.

12. ${}^{2n}C_n = (2n)!/(n!)^2$.

16. $\angle\text{FDE} = 180° - 2A$, $\angle\text{DEF} = 180° - 2B$, $\angle\text{DFE} = 180° - 2C$,
$\text{EF} = a\cos A$, $\text{FD} = b\cos B$, $\text{DE} = c\cos C$.

20. $\theta = \pi(6n \pm 1)/9$; $x = \cos 20° = 0{\cdot}93969$, $x = \cos 100° = -0{\cdot}17365$,
$x = \cos 140° = -0{\cdot}76604$.

23. $\angle X = 90° - \frac{1}{2}A$, $\angle Y = 90° - \frac{1}{2}B$, $\angle Z = 90° - \frac{1}{2}C$.

24. $$\left(-\frac{\{pl^2 - pm^2 + 2qlm + 2ln\}}{l^2+m^2}, -\frac{\{qm^2 - ql^2 + 2plm + 2mn\}}{l^2+m^2}\right);$$
$x + 7y = 13$.

25. $183y = 69x + 151$, $183y = 69x - 199$.

26. $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$;
$(\frac{1}{2}[x_1+x_2+y_1-y_2], \frac{1}{2}[y_1+y_2-x_1+x_2])$,
$(\frac{1}{2}[x_1+x_2-y_1+y_2], \frac{1}{2}[y_1+y_2+x_1-x_2])$.

27. $$y - \tfrac{1}{2}(y_1+y_2) = \frac{-(x_1-x_2)}{y_1-y_2}\{x - \tfrac{1}{2}(x_1+x_2)\}.$$

28. $x^2 + y^2 - 5x + 5y = 0$.

CHAPTER II (p. 62)

1. Limit $= 2$; Series $2 + x^2/6 - x^4/360$.

3. $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

4. $\text{Log}_e(1-3x+x^2)$ series valid for $\frac{1}{2}(3+\sqrt{5}) < x \leqslant \frac{1}{2}(3+\sqrt{13})$,
$\frac{1}{2}(3-\sqrt{13}) \leqslant x < \frac{1}{2}(3-\sqrt{5})$; Series $-\{3x + 7x^2/2 + 6x^3 + 47x^4/4\}$.

5. Coefficient of x^n is $-\frac{1}{n}(a^n + b^n)$; $\frac{(-1)^{n+1}}{n}\cos n\theta$.

Hint.—$\log_e\{1 - (a+b)x + abx^2\} = \log_e(1 - ax) + \log_e(1 - bx)$.

7. $\log_e 2 = 0{\cdot}693$, $\log_e 10 = 2{\cdot}303$. **8.** $1 + x + x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4$.

10. $-1/180$. **12.** $0{\cdot}076$ per cent. **14.** $0{\cdot}69315$. **15.** $4/5$.

16. $x = \log_e 13/9 = 0{\cdot}3677$, or $x = -\log_e 3 = -1{\cdot}0986$.

17. $\cosh x = \sec\theta$, $\tanh x = \sin\theta$. **18.** $\sinh u = \tan\theta$, $\sin\theta = \tanh u$.

19. $3x^2 + 3y^2 - 34x + 96 = 0$.

20. $z^2 = 2i$, $z^3 = -2 + 2i$, $\frac{1}{z} = \frac{1}{2} - \frac{1}{2}i$; Cube roots of i are $\pm\frac{1}{2}\sqrt{3} + \frac{1}{2}i$, $-i$.

21. (i) $\frac{3}{5} + i\,.\,\frac{4}{5}$; (ii) $\frac{1}{\sqrt{2}} + i\,.\,\frac{1}{\sqrt{2}}$; (iii) $\frac{1}{\sqrt{2}}\{\cosh\frac{1}{2} - i\sinh\frac{1}{2}\}$;

lengths of semi-axes $1/r + r$, $1/r - r$; $x^2 - y^2 = 2$.

24. (i) $p = 0 + i\,.\,5$, $q = 4/5 + i\,.\,3/5$.

(ii) $x = \dfrac{a(7 + 3\cos\theta)(3 + \cos\theta)}{2(5 + 3\cos\theta)}$, $y = \dfrac{3a\sin\theta(1 + \cos\theta)}{2(5 + 3\cos\theta)}$.

26. (i) line $y = 1$, (ii) circle $x^2 + y^2 = 4k^2$,

(iii) circle $\sqrt{3}(x^2 + y^2 - 4) = 4y$.

30. Straight line $x = 2$ (from $y = -5/4$ to $y = -3/4$). **31.** Modulus $= 4$.

33. $2y = \{(x^2 - 1) + y^2\}\tan\alpha$. **34.** Factors $-(b - c)(c - a)(a - b)$.

35. $256\cos^8\theta - 448\cos^6\theta + 240\cos^4\theta - 40\cos^2\theta + 1$.

36. $1 - 32\sin^2\theta + 160\sin^4\theta - 256\sin^6\theta + 128\sin^8\theta$.

37. $x = \dfrac{\tanh u\sec^2 v}{1 + \tanh^2 u\tan^2 v}$, $y = \dfrac{\tan v\,\mathrm{sech}^2 u}{1 + \tanh^2 u\tan^2 v}$; $u = \tanh^{-1}\frac{1}{2}(3 - \sqrt{5})$,

$v = \tan^{-1}\frac{1}{2}(\sqrt{5} + 1)$.

38. $u = \dfrac{\tan\frac{1}{2}x\,\mathrm{sech}^2\frac{1}{2}y}{1 + \tan^2\frac{1}{2}x\tanh^2\frac{1}{2}y}$, $v = \dfrac{\tanh\frac{1}{2}y\sec^2\frac{1}{2}x}{1 + \tan^2\frac{1}{2}x\tanh^2\frac{1}{2}y}$.

39. $\cosh v = \cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta$, $\cos u = \cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta$.

40. $p = \cos\theta$, $q = \sin\theta$; coefficient of x is $\sin\theta$.

41. Locus of P is $x^2 - y^2 = 2$; value $= 3 \pm i\sqrt{7}$.

43. $x = \cos u\cosh v$, $y = -\sin u\sinh v$; $u = 2n\pi \pm \frac{1}{3}\pi$, $v = \pm\log_e 3$.

45. $x = \cosh u\cos v$, $y = \sinh u\sin v$; $\dfrac{x^2}{\cosh^2 u} + \dfrac{y^2}{\sinh^2 u} = 1$, u constant;

$\dfrac{x^2}{\cos^2 v} - \dfrac{y^2}{\sin^2 v} = 1$, v constant.

46. $\frac{1}{2}\log_e(x^2 + y^2) + i(2n\pi + \alpha)$, where $\cos\alpha = x/\sqrt{(x^2 + y^2)}$, $\sin\alpha = y/\sqrt{(x^2 + y^2)}$,

$0 \leqslant \alpha \leqslant 360°$; $u = \frac{1}{2}\log_e\dfrac{(x - a)^2 + y^2}{(x + a)^2 + y^2}$, $v = 2n\pi + \beta - \gamma$,

where $\cos\beta = \dfrac{x - a}{\sqrt{\{(x - a)^2 + y^2\}}}$, $\sin\beta = \dfrac{y}{\sqrt{\{(x - a)^2 + y^2\}}}$,

$\cos\gamma = \dfrac{x + a}{\sqrt{\{(x + a)^2 + y^2\}}}$, $\sin\gamma = \dfrac{y}{\sqrt{\{(x + a)^2 + y^2\}}}$, $0 \leqslant \beta \leqslant 360°$, $0 \leqslant \gamma \leqslant 360°$.

48. $\prod_{k=0}^{k=n-1}\left\{x^2 - 2x\cos\frac{2k+1}{2n}\pi + 1\right\}$.

49. (i) $x = -\frac{1}{2}\left\{1 + i\cot\frac{2k+1}{16}\pi\right\}$, where $k = 0, 1, 2, \ldots, 7$.

CHAPTER III (p. 85)

1. $\dfrac{17}{27(1+x)} + \dfrac{10}{9(1+x)^2} - \dfrac{68}{27(1+4x)} + \dfrac{16}{9(1+4x)^2}$;

   (i) Valid for $-\frac{1}{4} < x < \frac{1}{4}$, coeff. of $x^{2n} = \frac{1}{27}[47 + 60n + 4^{2n}(96n - 20)]$,
   coeff. of $x^{2n+1} = -\frac{1}{27}[77 + 60n + 4^{2n+1}(96n + 28)]$.

   (ii) Valid for $x > 1$, or $x < -1$,
   coefficient of $x^{-2n} = \frac{1}{27}[47 - 60n - (24n + 5) \cdot 4^{-(2n-1)}]$,
   coefficient of $x^{-(2n+1)} = \frac{1}{27}[60n - 17 + (24n + 17)4^{-2n}]$.

2. $\dfrac{1}{8(1+x)} + \dfrac{1}{8(1-x)} + \dfrac{1}{4(1+x)^2} + \dfrac{1}{2(1+x)^3}$,

3. $\dfrac{25}{(5x-1)^2} - \dfrac{1}{2x-1}$;

   (i) $-1/5 < x < 1/5$, coefficient of $x^n = (n+1)5^{n+2} + 2^n$;

   (ii) $|x| > \frac{1}{2}$, coefficient of $1/x^n = (n-1)5^{2-n} - 1/2^n$;

   (iii) $1/5 < x < \frac{1}{2}$, $-1/5 > x > -\frac{1}{2}$,
   coefficient of x^n is 2^n or $(n+1) \cdot 5^{n+2}$,
   coefficient of x^{-n} is $(n-1)5^{-n+2}$ or $1/2^n$.

4. $\dfrac{x}{1 - x + x^2} - \dfrac{1}{2+x}$;

   coefficient of $x^n = (-1)^{n+1} \cdot \dfrac{1}{2^{n+1}}$ when n is a multiple of 3,

   $= (-1)^{n+1} \cdot \dfrac{1}{2^{n+1}} + 1$ when $n = 1, 2, 7, 8$, etc.,

   $= (-1)^{n+1} \cdot \dfrac{1}{2^{n+1}} - 1$ when $n = 4, 5, 10, 11$, etc.;

   series valid for $-1 < x < 1$.

6. (i) $\frac{1}{2}e + e^{-1} - 1$; (ii) $\frac{1}{4}\log_e 27/16$.

7. $2\left\{\dfrac{x-1}{x+1} + \frac{1}{3}\left(\dfrac{x-1}{x+1}\right)^3 + \ldots\right\}$, valid for $x > 0$.

8. (i) $\frac{1}{2}\left\{\dfrac{e^\theta + 1 - e^{(n+1)\theta} - e^{-n\theta}}{1 - e^\theta}\right.$

   (ii) $\dfrac{x\sin\theta - x^3\sin\theta - x^{n+1}\sin(n+1)\theta + 2x^{n+2}\sin xn\theta - x^{n+3}\sin(n-1)\theta}{(1 - 2x\cos\theta + x^2)^2}$
   $- \dfrac{n\{x^{n+1}\sin(n+1)\theta - x^{n+2}\sin n\theta\}}{1 - 2x\cos\theta + x^2}$.

9. (ii) $e^{\cos^2\theta}\sin(\frac{1}{2}\sin 2\theta)$.

10. (i) $\dfrac{1}{12}-\dfrac{1}{4(2n+1)(2n+3)}, \dfrac{1}{12}$; (ii) $12e-1$, $e^{\cos\theta}\sin(\sin\theta)$.

11. (i) 1; (ii) $e^{\cos\theta}\{\cos(\sin\theta)-\cos(\sin\theta-\theta)\}+\cos\theta$;
$\cos[\alpha+\frac{1}{2}(n-1)\beta]\sin\frac{1}{2}n\beta/\sin\frac{1}{2}\beta$. [*Hint.*—Multiply series by $\sin\frac{1}{2}\beta$.]

12. (i) $\frac{3}{4}\{3\sin\theta-\dfrac{1}{3^{n-1}}\sin(3^n\theta)\}$, $\frac{9}{4}\sin\theta$; (ii) $\cos\frac{1}{2}\theta/\sqrt{(2\cos\theta)}$.

13. (ii) $\cos(\sin\theta)\cosh(\cos\theta)$.

15. (i) 7/36; (ii) $3(e-1)$; (iii) $e^{\cos\theta}\cos(\sin\theta-\theta)-\cos\theta-1$.

16. (i) $\dfrac{2}{r+3}-\dfrac{1}{2(r+2)}-\dfrac{3}{2(r+4)}$, sum $=\dfrac{5}{24}-\dfrac{2n+5}{2(n+3)(n+4)}$;
(ii) $\frac{1}{3}\log_e 2-\frac{1}{2}\log_e 8/9=\log_e 3-4/3\log_e 2$.

CHAPTER IV (p. 119)

1. (*a*) $\dfrac{-15}{(3x-2)^6}$; (*b*) $\dfrac{-1}{\sqrt{(1-2t)}}$; (*c*) $-\frac{2}{3}(3s^5)^{-1/3}-0{\cdot}2s^{-1{\cdot}2}$;
(*d*) $-2/(3e^{2x})$; (*e*) $(y+1)^2(2-y)e^{-y+1}$; (*f*) $(2\log_e 3).3^{2z+3}+5/(1-z)$;
(*g*) $x-\dfrac{x^2+2}{\sqrt{(x^2+4)}}$; (*h*) $\dfrac{z-1}{z(1-2z)}$; (*i*) $\dfrac{\pi}{60}\cos\dfrac{\pi\theta}{60}=\dfrac{\pi}{60}\cos 30^\circ$;
(*j*) $-6\sin 4\theta$; (*k*) $\operatorname{cosec}\phi$; (*l*) $\sin^{-1}(2t-1)+\dfrac{t}{\sqrt{(t-t^2)}}$;
(*m*) $\dfrac{2-p^2}{\{(p+1)(p+2)\}^2}$; (*n*) $\dfrac{1}{\cos^{-1}3y}+\dfrac{3y}{\{\sqrt{(1-9y^2)}\}(\cos^{-1}3y)^2}$; (*o*) $-2\sin 2x$;
(*p*) $\sec^2\phi\tan\phi(2+3\tan\phi)$; (*q*) $\dfrac{2}{1+y^2}$; (*r*) $\dfrac{2}{z\sqrt{(z^2-4)}}$;
(*s*) $\dfrac{-1}{2\sqrt{\{(2-s)(1-s)\}}}$; (*t*) $\dfrac{12}{16-y^2}$; (*u*) $\dfrac{1}{x^2\cosh^{-1}x}\left\{2x\cosh^{-1}x\pm\dfrac{x^3}{\sqrt{(x^2-1)}}\right\}$;
(*v*) $-\frac{3}{2}\sqrt{\operatorname{sech}30}\,.\tanh 3\theta$; (*w*) $z^2e^{3z}\operatorname{cosech}z\left\{\dfrac{2}{z}+3-\coth z\right\}$;
(*x*) $\dfrac{4(2x-1)}{3(1-2x+2x^2)}$; (*y*) $2\operatorname{sech}^2\theta\{\operatorname{sech}^2 2\theta-\tanh\theta\tanh 2\theta\}$;
(*z*) $-\frac{1}{2}\operatorname{cosec}\frac{1}{2}x$.

2. (i) $\dfrac{1}{x^{\frac{1}{2}}(1-x)}, \pm\dfrac{1}{x\sqrt{(1-x^2)}}$; (ii) $-\frac{3}{2}$.

3. $-\operatorname{cosec}^2 x$. **4.** $\sec x\tan x$; $x=2$ or $38/9$.

5. (i) $\dfrac{2a^2x}{a^4+x^4}$; (ii) $\dfrac{x^2-1}{x^2-4}$.

6. $\log_e a$; $\frac{1}{2}a$; 0; $\dfrac{d}{dx}\left(\tanh^{-1}\dfrac{x}{a}\right) = -\dfrac{d}{dx}\left(\log_e \dfrac{a-x}{a+x}\right)$.

7. $\cos x - x \sin x$; (i) $\dfrac{3}{2(1+x)\sqrt{x}}$; (ii) $x^{a-1}\{a \log_e x + 1\}$; $\frac{1}{6}$.

8. (i) $\dfrac{3}{2b^2}(b^2 - x^2)$; (ii) $\sec x$; deduction: $\log_e \{\tan(\frac{1}{4}\pi + \frac{1}{2}x)\}$
$= 2\tanh^{-1}(\tan \frac{1}{2}x) + C$.

12. (ii) $\dfrac{dy}{dx} = \cot \dfrac{a+b}{2b}t$, $\dfrac{d^2y}{dx^2} = -\dfrac{(a+b)}{4ab}\operatorname{cosec}^3 \dfrac{a+b}{2b}t \operatorname{cosec} \dfrac{a-b}{2b}t$.

14. (i) $\dfrac{n!}{2}\left\{\dfrac{1}{(1-x)^{n+1}} - \dfrac{(-1)^n}{(1+x)^{n+1}}\right\}$.

16. $(a^2 + x^2)y_{n+2} + x(2n+1)y_{n+1} + n^2y_n = 0,\ n > 0$.

$$(\log_e a)^2 + \frac{2x}{a}\log_e a + \frac{x^2}{a^2} - \frac{x^3}{3a^3}\log_e a - \frac{x^4}{3a^4} + \dots .$$

17. $(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$.

19. $1 + \dfrac{\alpha}{2}x^2 + \dfrac{\alpha}{3}x^3 + \dfrac{1}{4}\left(\alpha + \dfrac{\alpha^2}{2}\right)x^4 + \dfrac{1}{5}\left(\alpha + \dfrac{5\alpha^2}{6}\right)x^5 + \dots .$

20. $x - \dfrac{x^3}{3} - \dfrac{2x^5}{15} - \dfrac{8}{105}x^7$. 22. $xy_{n+1} + (n - m)y_n = 0$.

23. $(1 - x^2)y_{n+1} - 2(r + nx)y_n - n(n-1)y_{n-1} = 0$;

Series $\sum\limits_{1}^{\infty} A_nx^n$, where $(n+1)A_{n+1} - 2rA_n - (n-1)A_{n-1} = 0$.

CHAPTER V (p. 158)

1. $3\sqrt[3]{\dfrac{s}{2}} + C$. 2. $\dfrac{4}{3}\sqrt{y^3} - \dfrac{25}{11}y^{2\cdot 2} + \dfrac{5}{2}y^{0\cdot 8} + \dfrac{1}{3}y^3 + 3y + C$.

3. $\dfrac{1}{2(2-3z)^2} + C$. 4. $\dfrac{1}{10}(2x+5)^5 + C$. 5. 2.

$-\dfrac{2}{3}\log_e(1 - 3p) + C$. 7. $\dfrac{2}{3}e^{3x+2} + C$. 8. $\dfrac{-2}{3e^{0\cdot 3y}}$

9. $2^{3z}/(3\log_e 2) + C$. 10. $\frac{1}{3}\cos(2 - 3\theta) + C$. 11. $3\sqrt{3}/2$.

12. $\pi/12$. 13. $\frac{1}{8}\log_e 3$. 14. $\frac{1}{2}\pi$. 15. $\frac{1}{2}\sinh^{-1} 2z/3 + C$.

16. $\frac{1}{3}\cosh^{-1} 3s + C$. 17. $\frac{1}{2}\log_e(4t^2 + 2t - 1) + C$.

18. $\log_e(\sin x + \cosh x) + C$. 19. $\frac{2}{3}\sqrt{(2p^3 - 3p + 2)} + C$.

20. $x - \frac{1}{3}\log_e(x+1) + \frac{4}{3}\log_e(x-2) + C$. 21. $\dfrac{\pi}{4} - \dfrac{2}{3}$.

22. $\frac{1}{3}\tan^{-1}\frac{1}{3}(z+1)+C.$ **23.** $\sinh^{-1}\frac{1}{3}(y+1)+C.$

24. $2+\log_e 5 = 3{\cdot}609.$ **25.** $3\tan^{-1}(p+2)+\frac{1}{2}\log_e(p^2+4p+5)+C.$

26. $3\sinh^{-1}(u+2)+\sqrt{(u^2+4u+5)}+C.$ **27.** $C-\frac{1}{[illegible]}\left\{\frac{1}{[illegible]}\tan^{-1}\frac{x}{[illegible]}+\frac{1}{[illegible]}\right\}.$

28. $\frac{1}{12}\left\{\log_e 2+\frac{\sqrt{3}}{3}\pi\right\}=0{\cdot}2089.$ **29.** $\frac{1}{2}\left\{\log_e(a^2+y^2)+\frac{a^2}{a^2+y^2}\right\}+C.$

30. $\frac{1}{4}\sqrt{(4x^2-4x-3)}-\frac{5}{4}\cosh^{-1}\frac{1}{2}(2x-1)+C.$

31. $C-\frac{2}{9}\sqrt{(18p-9p^2-5)}+\frac{7}{3}\sin^{-1}\frac{3(p-1)}{2}.$

32. $\frac{1}{2(z-1)}+\frac{1}{3}\log_e(z-2)-\frac{1}{4}\log_e(z-1)-\frac{1}{12}\log_e(z+1)+C.$

33. $\tan^{-1}u-\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}}+C.$

34. $\frac{1}{2}\{\sinh^{-1}(x+3)+(x+3)\sqrt{(10+6x+x^2)}+C\}.$

35. $\frac{1}{2}(v-1)\sqrt{(4v^2-8v-5)}-\frac{9}{4}\cosh^{-1}\frac{2}{3}(v-1)+C.$ **36.** $0.$

37. $\frac{1}{2}.$ **38.** $\frac{2}{3}\sin^3\frac{1}{2}x+C.$ **39.** $\frac{1}{5}\log_e\{\tan(\frac{1}{2}u+26^\circ\,34')\}+C.$

40. $\frac{1}{169}\tan(t+22^\circ\,37')+C.$ **41.** $\pi/12.$ **42.** $\frac{1}{3}\log_e\frac{3+\tan\frac{1}{2}v}{3-\tan\frac{1}{2}v}+C.$

43. $\frac{\pi}{6}+\frac{\sqrt{3}}{16}=0{\cdot}6319.$ **44.** $C-\frac{1}{10n}(\cos 5n\theta+5\cos n\theta).$

45. $\frac{1}{42n}\{3\sin 7nt+7\sin 3nt\}+C.$ **46.** $\frac{1}{24}\{6\sin 2\phi-\sin 12\phi\}+C.$

47. $\frac{8}{15}.$ **48.** $\frac{2}{35}.$ **49.** $\frac{1}{64}(20t-16\sin 2t+3\sin 4t)+\frac{1}{48}\sin^3 2t+C.$

50. $\frac{1}{192}\{12\theta+3\sin 2\theta-3\sin 4\theta-\sin 6\theta\}+C.$ **51.** $0.$

52. $\frac{1}{4}\pi-\frac{2}{3}=0{\cdot}1187.$ **53.** $\log_e(\cosh v)-\frac{1}{2}\tanh^2 v+C.$

54. $\frac{1}{32}(\sinh 4t-4t)+C.$ **55.** $C-\cot\theta-\frac{1}{3}\cot^3\theta.$

56. $\frac{1}{8}(4\phi+\sinh 4\phi)+C.$ **57.** $\frac{1}{2}\{y^2+y\sqrt{(y^2+1)}+\sinh^{-1}y\}+C.$

58. $\sin^{-1}(x-2)+\sqrt{(4x-x^2-3)}+C.$ **59.** $\frac{1}{3(a^2-z^2)^{3/2}}+C.$

60. $\frac{p}{a^2(a^2+p^2)^{\frac{1}{2}}}+C.$ **61.** $\frac{3}{16}\pi a^4.$ **62.** $C-\frac{2\sqrt{(1-3x)}}{405}(8+12x+27x^2).$

63. $\pi a.$ **64.** $\pi.$ **65.** $\frac{2}{35}.$ **66.** $\frac{1}{2}\theta-\frac{1}{2\sqrt{3}}\log_e\frac{\sqrt{3}+\tan\frac{1}{2}\theta}{\sqrt{3}-\tan\frac{1}{2}\theta}+C.$

67. $\sec^{-1}x+C.$ **68.** $y\sinh y-\cosh y+C.$

69. $\frac{\theta^2}{4}-\frac{\theta}{8}\sin 4\theta-\frac{1}{32}\cos 4\theta+C.$ **70.** $x\sin^{-1}x+\sqrt{(1-x^2)}+C.$

71. $(v^4/16)(4\log_e v - 1) + C$. **72.** $C - \frac{1}{13}e^{-2\theta}(3\cos 3\theta + 2\sin 3\theta)$.

73. $\frac{2}{15}(1+x)^{3/2}(3x-2) + C$. **74.** $C - 2e^{-u/2}(u^2 + 4u + 8)$.

75. $\frac{5}{32}\pi + \frac{1}{12}$. **76.** $x\sec^{-1}x - \cosh^{-1}x + C$.

77. $\frac{1}{2}x^2\log_e(1+x) + \frac{1}{2}x - \frac{1}{4}x^2 - \frac{1}{2}\log_e(1+x) + C$.

78. $\frac{1}{2}\log_e\{\sinh(2\theta+1)\} + C$. **79.** $\frac{1}{2}\theta^2\sin 2\theta + \frac{1}{2}\theta\cos 2\theta - \frac{1}{4}\sin 2\theta + C$.

80. $C - \{\frac{1}{2}\sin^2\theta + \sin\theta + \log_e(1-\sin\theta)\}$. **81.** $u - \frac{1}{2}\tanh 2u + C$.

82. $\frac{1}{2}\operatorname{cosech}^{-1}2x + C$. **83.** $\frac{7}{15}$.

84. $\frac{1}{2}\log_e 2 - \frac{1}{2} = -0{\cdot}1534$. **85.** (i) 0; (ii) $\pi^3/48 + \pi/8$.

86. (i) $x + \log_e(x^2 - x + 1) + \dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}} + C$; (ii) $2x^{\frac{1}{2}} + \log_e\dfrac{x^{\frac{1}{2}}-1}{x^{\frac{1}{2}}+1} + C$.

(iii) $2\pi/\sqrt{5} - \frac{1}{2}\log_e 5$.

87. (i) $\log_e\{(1+x)/x\} - 1/x + C$.

88. $x - \log_e(x^2 + x + 1) + \dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}} + C$; $\pi - 2$;

$2y + a^2/y$, $1/y$, $\int y\,dx = \frac{1}{2}xy - a^2\log_e(x+y) + C$

$= \frac{1}{2}[x\sqrt{(x^2-a^2)} - a^2\log_e\{x + \sqrt{(x^2-a^2)}\}] + C$.

89. (i) $2\log_e\dfrac{\sqrt{(x+1)}-1}{\sqrt{(x+1)}+1} - 6\sqrt{(1+x)} + C$; (ii) $\dfrac{e^x}{1+x} + C$; (iii) $5\pi/2$.

90. (i) $\dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C$; (ii) $e^{a\sin^{-1}x}\cdot\dfrac{x + a\sqrt{(1-x^2)}}{a^2+1} + C$;

(iii) $\dfrac{x^4-1}{4}\tan^{-1}x - \dfrac{x}{12}(x^2-3) + C$.

91. (i) $\frac{1}{3}\log_e\dfrac{1-\sqrt{x}+x}{1+2\sqrt{x}+x} + \dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2\sqrt{x}-1}{\sqrt{3}} + C$;

(ii) $\dfrac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\left(\sqrt{\dfrac{a-b}{a+b}}\tan\frac{1}{2}x\right) + C$; $a^2(\frac{1}{4}\pi - \frac{1}{2})$.

92. $2e^{\sqrt{x}}(\sqrt{x}-1) + C$; $\frac{1}{2}e^{\sin^{-1}x}\{x + \sqrt{(1-x^2)}\} + C$; $\dfrac{1}{\sin\alpha}\log_e\dfrac{\cos\frac{1}{2}(\theta-\alpha)}{\cos\frac{1}{2}(\theta+\alpha)} + C$.

93. (i) $\log_e\dfrac{1-x}{1+x} + \dfrac{1}{1-x} + C$; (ii) $\dfrac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\left(\sqrt{\dfrac{a-b}{a+b}}\tan\frac{1}{2}x\right) + C$;

(iii) $\dfrac{e^{-a\pi/2}}{a^2+b^2}\{b\sin\frac{1}{2}b\pi - a\cos\frac{1}{2}b\pi\} + \dfrac{a}{a^2+b^2}$.

94. (i) $\frac{1}{2}\pi - 1$; (ii) $\dfrac{\pi}{144} + \dfrac{\sqrt{3}}{8} - \dfrac{2}{9}$;

(iii) $\log_e(\tan\frac{1}{2}\theta) - \dfrac{2}{\sqrt{3}}\log_e\dfrac{\sqrt{3}+\tan\frac{1}{2}\theta-2}{\sqrt{3}-\tan\frac{1}{2}\theta+2} + C$.

95. (i) $-\dfrac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}} + C$; (ii) $x\tan\frac{1}{2}x + C$; (iii) $\frac{1}{2}\pi(b-a)$.

96. (i) $a\sin^{-1}(x/a) - 2\sqrt{(a^2 - x^2)} + C$;

(ii) $\frac{1}{5}\log_e \tan\frac{1}{2}(x + 36°\,52') + C = \frac{1}{5}\log_e \dfrac{1 + 3\tan\frac{1}{2}x}{3 - \tan\frac{1}{2}x} + C.$

(iii) $\dfrac{x^3}{3}\tan^{-1}x - \frac{1}{6}x^2 + \frac{1}{6}\log_e(1 + x^2) + C.$

97. (i) $\pi a^6/32$; (ii) $5/4 - 3\pi/8$; (iii) $\frac{1}{2}$.

98. (i) $b\log_e(1 + \sin\theta) + \theta - \dfrac{2(a - 1)}{1 + \tan\frac{1}{2}\theta} + C$; (ii) $\dfrac{x^3}{12(4 + x^2)^{3/2}} + C$;

(iii) $\frac{1}{2}[\log_e 2 - \frac{1}{2}] = 0{\cdot}0966.$

99. (i) $C - e^{-x}(x^2 + 2x + 2)$; (ii) $13/15 - \pi/4 = 0{\cdot}0813$;

(iii) $(\frac{2}{5} - \frac{6}{7}x + \frac{2}{3}x^2 - \frac{2}{11}x^3)x^{5/2} + C$; (iv) $\sin x/(2 + \cos x) + C.$

100. (i) $\frac{3}{8} - \frac{19}{8}e^{-2}$; (ii) $\frac{2}{5}$; (iii) $2\sinh^{-1}\frac{1}{2}(x+3) + \frac{1}{2}(x+3)\sqrt{(x^2 + 6x + 13)} + C.$

101. (i) $8/105$; (ii) $\dfrac{1}{e^2}\{\log_e(1 - e\cos x) - (1 - e\cos x)\} + C$; (iii) $\frac{3}{8} + 22\frac{5}{8}\,e^8.$

102. (i) $\frac{1}{4}\pi - \frac{1}{2}$; (ii) $2/35$; (iii) $\frac{1}{5}(3\pi/2 + \log_e 2).$

103. (i) $\tan^{-1}(x + 2)$; (ii) $\frac{1}{2}\log_e\dfrac{x + 1}{x + 3}$; (iii) $2\cos x + 2x\sin x - x^2\cos x$;

(iv) $\frac{1}{2}\tan^{-1}(\frac{1}{2}\tan\frac{1}{2}\theta).$

104. (i) $2\sinh^{-1}\sqrt{(x - 2)} - 2\sqrt{\dfrac{x - 1}{x - 2}}$; (ii) $x - \dfrac{4}{\sqrt{3}}\tan^{-1}\dfrac{2\tan\frac{1}{2}x + 1}{\sqrt{3}}$;

(iii) $\frac{1}{3}\theta(\cos^3\theta - 3\cos\theta) + \frac{1}{9}(6\sin\theta + \sin^3\theta)$; $\frac{1}{4}\pi + \frac{1}{2}\log_e 2 = 1{\cdot}13.$

105. $\sin^{-1}x + \sqrt{(1 - x^2)}$; $\frac{1}{2}\tan^{-1}(\frac{1}{2}\tan\frac{1}{2}\theta)$; $2\tan^{-1}(\tanh\frac{1}{2}u)$; $0{\cdot}209.$

106. (*a*) $\pm\cosh^{-1}\left(\dfrac{\cos\theta}{\cos\alpha}\right)$; (*b*) $\sin^{-1}\left(\dfrac{\sin\theta}{\sin\alpha}\right)$; (*c*) $\frac{9}{28} = 0{\cdot}321$;

(*d*) $\frac{1}{2}\{(\sqrt{2} + \log_e(\tan 3\pi/8)\} = 1{\cdot}15.$

107. (*a*) $\sqrt{2}\pi/4 = 1{\cdot}11$; (*b*) $(1/\sqrt{2})\log_e(\sqrt{2} + 1) = 0{\cdot}623$;

(*c*) $\frac{1}{4}\pi(2 - \sqrt{2}) = 0{\cdot}460$; (*d*) $\frac{1}{4}\pi = 0{\cdot}785.$

108. n odd: $I = \dfrac{(n - 1)(n - 3)\ldots 2}{n(n - 2)\ldots\ldots 3}$; n even: $I = \dfrac{(n - 1)(n - 3)\ldots 3\,.\,1}{n(n - 2)\ldots\ldots 4\,.\,2}\,\dfrac{\pi}{2}$

$\theta = \pi/4$; Area $= 19\pi a^2/256.$

109. $a^6\left\{\frac{7}{48}\sqrt{2} + \frac{1}{16}\log_e\left(\tan\dfrac{3\pi}{8}\right)\right\}.$

110. $-2/315.$ **111.** $n!\,q^n/\{(p + 1)(p + q + 1)\ldots(p + \overline{n - 1}\,.\,q + 1)(p + nq + 1)\}.$

112. (i) 4π; (ii) $(\pi/64 - 1/48)/a^3.$ **113.** $8/315.$ **114.** $\frac{3}{10}\{e^{\pi/2} + e^{-\pi/2}\} = 1{\cdot}5055.$

115. $m^n\,.\,n!/(mn + 1)\{m(n - 1) + 1\}\ldots\{m + 1\}.$

116. $u_4 = \dfrac{3\pi^2}{64} - \dfrac{1}{4}$, $u_5 = \dfrac{4\pi}{15} - \dfrac{149}{225}.$ **117.** $33{\cdot}6 - \frac{40}{3}\sqrt{5} = 3{\cdot}784.$

118. $\dfrac{(2m-1)(2m-3)\ldots 3.1}{(2m+2)\,2m(2m-2)\ldots 2}\cdot\dfrac{\pi}{2}$. **119.** $\dfrac{-746}{35}a^7 + \dfrac{432}{35}a^7\sqrt{3}$.

120. $\frac{3}{2} - 15\pi/32$. **122.** $2n^{p+3}/(p+1)(p+2)(p+3)$. **123.** $a = 1/12$.

125. If $I_{2n} = \int \cos^{2n}\theta\, d\theta$, then $2nI_{2n} = \cos^{2n-1}\theta \sin\theta + (2n-1)I_{2n-2}$;

$$\frac{(2n)!\, r\pi}{2^{2n+1}(n!)^2}.$$

CHAPTER VI (p. 181)

1. $a = 3/4,\ b = -3/20,\ c = 1/60$.

4. $2x^{1/2} + \frac{1}{3}x^{3/2} + \frac{3}{20}x^{5/2} + \ldots + \dfrac{1.3.5\ldots(2n-1)}{2^{n-1}(2n+1)n!}x^{n+\frac{1}{2}} + \ldots$; 0·2003.

7. $1/\sqrt{(1+x^2)}$. **10.** $(-1)^{n+1}x^n/(1+x)$; $(-1)^n x^n/n(1+x)^2$.

12. $x = \frac{2}{3}$ for inflexion; $y = 0\,(x=1)$, $y = \dfrac{24(6 \pm \sqrt{33})}{17 \pm \sqrt{33}}$ [for $x = \frac{1}{2}(5 \pm \sqrt{33})$].

13. Maximum 8/3, minimum 0. **14.** h.

16. Time $= \dfrac{\sqrt{(a^2+x^2)}}{u} + \dfrac{\sqrt{\{b^2+(c-x)^2\}}}{v}$. **18.** $a^{2/3}b^{2/3}/\sqrt{(a^{2/3}+b^{2/3})}$.

19. $\sin^{-1}(\sqrt{2}-1) = 24°\ 28'$. **20.** $l = R/\sqrt{3}$, $V = 2\sqrt{3}\,.\,\pi R^3/27$.

22. $4\pi r^3 + \pi r^2 - 150 = 0$, where r = the radius. **23.** $\pi/8,\ 3\pi/8$. **24.** ae.

25. (a) $1/n$; (b) $\frac{1}{2}\{2\log_e a + 1\}$; (c) $\frac{1}{2}\{2\log_e a - 1\}$.

26. (a) $(3/2b^2)(b^2 - x^2)$; (b) $1/(2\cos^2\alpha)$.

CHAPTER VII (p. 219)

2. $y = \pm\dfrac{2\sqrt{3}}{9}x$. **4.** $y = a^2(3x - 2a)$. **6.** $2a\cos\theta$. **10.** 1 : 2.

11. Radius of curvature $= a\cosh^2 x/a = y^2/a$; $(x - \frac{1}{2}a\sinh 2x/a,\ 2y)$.

12. $PI = r\operatorname{cosec}\alpha$; $r = a\cot\alpha\, e^{(\theta-\pi/2)\cot\alpha}$, radius of curvature $= OI\operatorname{cosec}\alpha$.

15. $\rho = -4a\sin\frac{1}{2}t$; $\{a(t+\sin t),\ a(\cos t - 1)\}$.

18. (i) $2\{x^2\tan^{-1}y/x - y^2\tan^{-1}x/y\}$.

20. (i) $\dfrac{dy}{dx} = \dfrac{4x^3 - 5a^2y}{5a^2x - 4y^3}$, $\dfrac{d^2y}{dx^2} = \dfrac{10a^2xy(16x^2y^2 + 75a^4)}{(5a^2x - 4y^3)^3}$. (ii) $\dfrac{\partial^2 z}{\partial v\,\partial u} = 0$.

23. Error $\delta a = \dfrac{(b - c\cos A)\,\delta b + (c - b\cos A)\,\delta c}{\sqrt{(b^2 + c^2 - 2bc\cos A)}}$;

For minimizing δb make $\angle C = 90°$.

28. (i) $\dfrac{\partial u}{\partial x} = \cos x \cosh y - \dfrac{\sin x \sinh y}{2x + 2y + 1}$; (ii) $r^2 \dfrac{\partial^2 V}{\partial r^2} + r\dfrac{\partial V}{\partial r} + \dfrac{\partial^2 V}{\partial \theta^2} = 0.$

29. $a = -3.$

30. (i) $\mu = -3/2.$ (ii) $\dfrac{\partial V}{\partial x} = ye^{xv}\dfrac{\partial V}{\partial u} + 2x\dfrac{\partial V}{\partial v}, \dfrac{\partial V}{\partial y} = xe^{xv}\dfrac{\partial V}{\partial u} + 2y\dfrac{\partial V}{\partial v}.$

CHAPTER VIII (p. 240)

1. (*a*) 0; (*b*) 950; (*c*) −10,813; (*d*) 1; (*e*) −475; (*f*) −232; (*g*) 164.

3. $x = 0, y = \frac{1}{3}, z = 0.$ **4.** Value = −8.

5. $x = -(a + b + c)$ or $\pm\sqrt{(a^2 + b^2 + c^2 - bc - ca - ab)}$;
$(a - b)(b - c)(c - a)(abc + 1).$

6. (i) $x = \frac{5}{3}$ or $-\frac{1}{6}$.

7. (i) $x = -\frac{1}{2}a$ (three times); (ii) $2abc(a + b + c)^3$.

8. (i) $x = 3$ or $(1 \pm \sqrt{561})/10$. **9.** (ii) $4\,a^2b^2c^2$.

10. (i) $-2\cos^2\alpha + \sin^2\alpha + 2\sin\alpha\cos\alpha$;
(ii) $\lambda = 14$ or 3; when $\lambda = 3, x = \frac{1}{6}, y = \frac{3}{2}$; when $\lambda = 14, x = -\frac{1}{5}, y = \frac{2}{5}$.

11. (i) $x = -3$ or $\pm\sqrt{3}$.

13. (i) $(a - b)(b - c)(c - a)$; (ii) $x = \frac{5}{2}, y = 3, z = -4.$

14. $\lambda = 3$ or $-1/22$.

CHAPTER IX (p. 281)

1. Condition is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$;

Product of perpendiculars $= \dfrac{ap^2 + 2hpq + bq^2 + 2gp + 2fq + c}{\sqrt{(a^2 - 2ab + b^2 + 4h^2)}}.$

2. $\dfrac{x^2 - y^2}{a - b} = \dfrac{xy}{h}$; $3x + 2y + 4 = 0, 2x - 3y + 7 = 0.$

3. OA is $y = \dfrac{hg - af}{hf - bg}x$, BC is $2gx + 2fy + c = 0.$

4. Equation is $r^2(ax^2 + 2hxy + by^2) - r(2gx + 2fy)(px + qy) + c(px + qy)^2 = 0.$
Condition is $r^2(a + b) - 2r(gp + fq) + c(p^2 + q^2) = 0.$

5. $C \equiv \left\{\dfrac{2r(hq - bp)}{aq^2 - 2hpq + bp^2}, \dfrac{2r(hp - aq)}{aq^2 - 2hpq + bp^2}\right\}.$

6. $\lambda = 10$ or -2; $6\sqrt{2}$ units.

7. $x^2 + y^2 + 2\lambda(x + y) - (6a\lambda + 5a^2) = 0.$ Common points $(a, 2a)$, $(2a, a)$.

8. Circle $p^2x^2 + p^2y^2 - 2x - 2py + 1 = 0$; tangent $2py = (p^2 - 1)x$.

10. Centre $(a, 3a)$, radius $3a/5$.

11. $(c^2 - a^2)(l^2 + m^2) + 2cln + 2n^2 = 0$.

12. $\sqrt{\left(x'^2 + y'^2 + \frac{2g}{a}x' + \frac{2f}{a}y' + \frac{c}{a}\right)}$.

13. $\{0, \pm\sqrt{(-c)}\}$, with the given choice of circles.

15. $hx + ky = a^2$; $2(hx + ky) = a^2 + h^2 + k^2$.

16. $ty = x + at^2$; $y^2 = 4a(a - x)$.

17. $3y^2 = 4ax$. **18.** $\{at_1t_2, a(t_1 + t_2)\}$. **19.** $x = \frac{1}{2}(-6 \pm \sqrt{34})a$, $y = \pm a/\sqrt{2}$.

20. Tangent $yt = x + at^2$, normal $tx + y = 2at + at^3$; $2y^2 + ax = 0$.

21. $(9a, -6a)$, $(16a, 8a)$.

22. Normal $y + tx - 2at - at^3 = 0$; $x = a(t_1^2 + t_1t_2 + t_2^2 + 2)$, $y = -at_1t_2(t_1 + t_2)$.

23. Line is $y = -c^2/a$.

25. Find the images of S in RQ and l and let these be S_1, S_2. S_1S_2 is the directrix. Let l cut the directrix in L. Draw SP perpendicular to SL cutting l in P, which is the point of contact.

CHAPTER X (p. 324)

1. $m_1m_2 = -b^2/a^2$.

2. $x = \dfrac{ab^2 \cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$, $y = \dfrac{a^2b\sin\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$.

3. Locus is $a^2x^2 + b^2y^2 = a^2(a^2 - b^2)$.

6. Tangent $axx_1 + byy_1 = 1$; Normal $\dfrac{x - x_1}{ax_1} = \dfrac{y - y_1}{by_1}$.

7. Locus is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{1}{k}$, where $k = \dfrac{a^2}{OP^2} + \dfrac{b^2}{OQ^2}$.

9. $\dfrac{x}{a}(1 - \tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta) + \dfrac{y}{b}(\tan\frac{1}{2}\alpha + \tan\frac{1}{2}\beta) = 1 + \tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta$,

or $\dfrac{x}{a}\cos\frac{1}{2}(\alpha + \beta) + \dfrac{y}{b}\sin\frac{1}{2}(\alpha + \beta) = \cos\frac{1}{2}(\alpha - \beta)$.

15, 16, etc. $x + t_1t_2y = c(t_1 + t_2)$.

CHAPTER XI (p. 347)

5. $5y = 2x + 2,\ x + 2y + 1 = 0.$

6. Focus $(\frac{48}{25}, \frac{36}{25})$; directrix $4x + 3y = 36.$

7. $x \cos \alpha + y \sin \alpha = \pm c \sqrt{\cos 2\alpha}.$ $\quad x \cos \alpha + y \sin \alpha = \pm c \sqrt{\sin 2\alpha}.$

8. Asymptotes $x - 2y - 1 = 0,\ x + y - 4 = 0,$
   or $x^2 - xy - 2y^2 - 5x + 7y + 4 = 0;$
   centre $(3, 1)$; $x^2 - xy - 2y^2 - 5x + 7y - 2 = 0.$

10. $(x + y + 1)(px + qy + r) + k = 0$, k any constant.

11. $3x - 2y + 1 = 0,\ x + 7y - 3 = 0.$

CHAPTER XII (p. 381)

1. $\sqrt{2}(x + y + z - 10) + y - z = 0,\ \sqrt{2}(x + y + z - 10) - y + z = 0.$

2. $\dfrac{x-2}{9} = \dfrac{y+4}{-13} = \dfrac{z+2}{-24}.$ $\qquad$ 3. $\dfrac{x}{1} = \dfrac{y}{-1} = \dfrac{z}{2},\ \dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{-1}.$

4. $3x - 8y + 7z + 4 = 0,\ 3x + 2y + z = 0;$ $\dfrac{x+1}{11} = \dfrac{y-1}{-9} = \dfrac{z-1}{-15}.$

5. $5y - 3z = 12,\ 34x - 9y - 15z = 20;$ $EF = 8/\sqrt{34}.$

6. Plane is $4x - 2y + z = 0$, $(-\frac{1}{21}, \frac{11}{21}, \frac{26}{21})$.

7. Protection of line is $\dfrac{x+4}{5} = \dfrac{y}{3} = \dfrac{z-4}{-4}$; point $\left(\dfrac{-3}{2}, \dfrac{3}{2}, 2\right).$

8. $\dfrac{x}{1} = \dfrac{y}{\pm\sqrt{11}} = \dfrac{z}{2}.$ $\qquad$ 9. $\dfrac{x-3}{\sqrt{2} \pm 1} = \dfrac{y-2}{\sqrt{2} \mp 1} = \dfrac{z+3}{-\sqrt{2}}.$

10. Shortest distance $= 13$ units; direction cosines given by $l : m : n = 3 : 4 : 12.$

12. (i) $65^\circ\ 57\frac{1}{2}'$; (ii) $46^\circ\ 39\frac{1}{2}'$.

13. $6x + 3y - 2z = 18,\ 2x - 3y - 6z = 6.$

14. $\cos^{-1} \dfrac{10}{\sqrt{418}} = 60^\circ\ 43'$; $\dfrac{x}{14} = \dfrac{y}{-11} = \dfrac{z}{1}.$

15. $\left.\begin{array}{l} 9x + 13y - 24z = 137 \\ 19x + 34y - 31z + 548 = 0 \end{array}\right\}$; length $= 49/\sqrt{59}.$
    points $(\frac{811}{59}, \frac{-2138}{59}, \frac{-1709}{59})$,
    $(\frac{-468}{59}, \frac{-2285}{59}, \frac{-1750}{59})$.

16. $9x + 11y - 6z - 13 = 0$; Area $= \dfrac{169\sqrt{238}}{1188}$; $\left(\dfrac{117}{238}, \dfrac{143}{238}, -\dfrac{78}{238}\right).$

17. $\dfrac{x}{2} = \dfrac{y}{-1} = \dfrac{z}{-4}$; $\left(\dfrac{2}{3}, \dfrac{-1}{3}, \dfrac{-4}{3}\right).$

18. Common point is $(4, 2, -1)$; plane containing the lines is $x - y - z = 3$; other plane is $4x + 5y - z = 27$.

20. $$\begin{vmatrix} a - a_1 & b - b_1 & c - c_1 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} \Big/ \sqrt{\{(mn_1 - m_1 n)^2 + (nl_1 - n_1 l)^2 + (lm_1 - l_1 m)^2\}};$$
length $= 22$ units, equation $\dfrac{x-1}{6} = \dfrac{y+1}{6} = \dfrac{z-4}{7}$.

22. $(1, 2, 3)$; L' is $\dfrac{x-1}{0} = \dfrac{y-2}{1} = \dfrac{z-3}{-1}$; angle between L and L' is $30°$; direction cosines are $\dfrac{1}{\sqrt{2}}, 0, \dfrac{-1}{\sqrt{2}}$.

23. Line is $\dfrac{x-1}{4} = \dfrac{y-2}{1} = \dfrac{z-3}{-2}$; point is $\left(\dfrac{19}{7}, \dfrac{17}{7}, \dfrac{15}{7}\right)$; plane is $x - 2y + z = 0$.

24. $2x - 3y + 6z = 12a$, $2x - 3y + 6z + 2a = 0$; shortest distance $= 2a$.

26. $\dfrac{x - \dfrac{2ab^2}{b^2 + c^2}}{0} = \dfrac{y}{c} = \dfrac{z}{b}$; shortest distance $= 2bc/\sqrt{(b^2 + c^2)}$; volume $= \frac{2}{3}abc$.

27. Angle $90°$; plane $23x - 13y + 32z = 93$.

28. Plane $x + y - z = 0$; angle is $60°$.

29. Plane $x - 2y + z = 0$; perpendicular distance $= 2\sqrt{(6/29)}$.

CHAPTER XIII (p. 413)

3. Condition is $\{p - (l\alpha + m\beta + n\gamma)\}^2 = r^2(l^2 + m^2 + n^2)$;
$x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0$.

5. $(2, 2, 3)$; $5x^2 + 5y^2 + 5z^2 - 14x - 14y - 18z = 0$.

7. (i) $(2, 3, 6)$, $(\frac{4}{3}, \frac{5}{3}, \frac{20}{3})$; (ii) $\sqrt{6}$.

9. $x - 4y - 2z + 1 = 0$; $Q \equiv (\frac{11}{7}, \frac{-2}{7}, \frac{13}{7})$.

10. (ii) $21(x^2 + y^2 + z^2) + 58x - 10y + 20z + 26 = 0$.

11. $x^2 + y^2 + z^2 + 2x - 8z = 0$.

12. $(x_1 - \alpha)(x - x_1) + (y_1 - \beta)(y - y_1) + (z_1 - \gamma)(z - z_1) = 0$;
$4x^2 + 4y^2 + 4z^2 + 10x - 25y - 2z = 0$.

13. $\dfrac{\pi}{6}\{(2m^2 - 1)(a^2 + b^2 + c^2)\}^{3/2}$.

14. $2x^2 + 2y^2 + 2z^2 - 7x - 6y - 9z + 13 = 0$;
$z = 0$, $199x^2 + 212y^2 - 84xy - 1260x - 1080y + 828 = 0$.

15. $x^2 + y^2 + z^2 + 20x + 20y + 18z + 72 = 0$; $x^2 + y^2 + z^2 - 2z - 8 = 0$.

17. $\dfrac{l^2}{a} + \dfrac{m^2}{b} + \dfrac{n^2}{c} = p^2$; $3x + 2y = 10,\ 2y + 9z = 10$; $\dfrac{x-3}{3} = \dfrac{y-\frac{1}{2}}{0} = \dfrac{z}{-1}$.

20. $axx_1 + byy_1 + czz_1 = 1$; $\dfrac{x-2}{18} = \dfrac{y}{-9} = \dfrac{z+\frac{2}{9}}{2}$.

26. $a^2l^2 + b^2m^2 + c^2n^2 = p^2$.

29. $8x^2 + 13y^2 + 13z^2 - 8yz - 12zx - 12xy + 20x - 66y + 36z + 21 = 0$.

30. $\dfrac{x-\alpha}{\beta} = \dfrac{y-\beta}{\alpha} = \dfrac{z-\gamma}{-c}$.

31. $x^2 + 7y^2 + z^2 + 8yz - 16zx + 8xy = 0$. **32.** $109^\circ\ 28'$.

33. $\beta x + \alpha y - \gamma z + a = 0$; $\dfrac{x - a\alpha}{(\gamma \pm 1)^2} = \dfrac{y - a\beta}{2\beta^2} = \dfrac{z - a\gamma.}{2\beta(\gamma \pm 1)}$.

CHAPTER XIV (p. 438)

1. $\dfrac{24\sqrt{3}}{5}a^2$. **2.** $\frac{1}{2}[2\sqrt{5} + \log_e(2+\sqrt{5})]$. **5.** $\frac{3}{2}(\pi + 1)a^2$. **7.** $16a^2/15$.

8. Area $= 3\pi a^2/4$; volume $= \pi^2 a^3/4$ cubic units.

10. Length $= 6a$; surface $= 12\pi a^2/5$ square units. **13.** $2\pi^2 a^3$.

16. Volume $= 3\pi a^3/10$; area $= \dfrac{\pi a^2}{96}\{134\sqrt{5} - 16 - 3\log_e(2+\sqrt{5})\}$.

18. $\dfrac{20\pi a^3}{3}$. **19.** $\dfrac{2h}{3}[2b + a]$. **20.** $16\pi a^3/105$. **22.** $128\pi a^3/15$.

23. $\left(\dfrac{4a}{3\pi}, \dfrac{4b}{3\pi}\right)$; $\pi(10 - 3\pi)a^2b^2/6\sqrt{(a^2+b^2)}$. **25.** $\frac{1}{6}$ square unit; $(6{\cdot}4,\ 2{\cdot}5)$.

26. On the radius bisecting the central angle θ: $\dfrac{2r\sin\frac{1}{2}\theta}{\theta}$ for circular arc, $\dfrac{4r}{3\pi}$ for semicircular area, from centre;

(i) $2\pi a(2a + \pi b)(+4\pi ab$ for *total* area); (ii) $\frac{1}{3}\pi a^2(4a + 3\pi b)$.

28. $(\pi a,\ 4a/3)$. **29.** $\frac{3}{8}\sqrt{(R^2 - a^2)}$.

CHAPTER XV (p. 461)

1. $y = 2\sin^{-1}x + 2\sqrt{(1-x^2)} + C$. **2.** $y = \dfrac{\sqrt{x}}{5}(x^2 - 5) + C$.

3. $x = \log_e\{C(y-1)/y\}$. **4.** $\tan^{-1}y = x + \frac{1}{4}\pi$.

5. $x = \sqrt{(1+y^2)} + C$. **6.** $\sin y = Cxe^{x^2/2}$.

7. $y^2 = 1 - C/(1+x^2)^2$. **8.** $\sqrt{(y^2-1)} = x + \log_e x + C$.

9. $2(y - \tan^{-1}y) = \log_e Cx$. **10.** $y = \dfrac{1}{x}\log_e Cx$.

11. $4\log_e(6y - 3x + 1) = 6x - 3y + C$.

12. $y = 2(8x^2 - 4x - 1 + e^{-4x})$.

13. $y - x - \log_e \frac{1}{2}(x + y) = 2$.

14. $2x + \log_e \dfrac{1 + y - x}{1 - y + x} = 0$.

15. $y = x^2 - 2x + 2 + Ce^{-x}$.

16. $y = \frac{2}{3}(\sin^2 x - \operatorname{cosec} x)$.

17. $y = e^{-3x}\{(1 + x)^3 - 1\}$.

18. $y = x + 1 + C/x^3$.

19. $y^2 = Ce^{2x} - 4x - 2$.

20. (i) $y = (1 - x^2)^{1/2}\{\frac{1}{2}a(\sin^{-1} x)^2 + C\}$; (ii) $y = \frac{1}{2}(x^2 - 1) + Ce^{-x^2}$;
(iii) $(x + y)(y - x + 2)^3 = C$.

21. (i) $a = 3$; (ii) $x - y + 1 = 3\log_e(x + y)$.

22. (i) $y = \frac{1}{2}\left\{\left(\sqrt{\dfrac{1 + x}{1 - x}}\right)\sin^{-1} x + x(1 + x)\right\} + C\sqrt{\dfrac{1 + x}{1 - x}}$;
(ii) $(y - x)^3 = Cx^2y^2$; (iii) $y = \dfrac{2(x + 1)}{x}\log_e(x + 1) + \dfrac{2}{x} + \dfrac{C(x + 1)}{x}$.

23. (i) $y = \cos x + C\cos^2 x$; (ii) $(y + 2x)^3(y + x)^2 = C$.

24. (i) $y = (1 - x^2)\log_e \dfrac{C}{(1 - x^2)^{1/2}}$; (ii) $(y - x)^2 = C(y - 2x)$.
(iii) $y^2 = \dfrac{1}{x}(C - \cos x)$.

25. $y^2 = x^3(Cx^3 - 2)$.

26. (*a*) $(x - y)^4(x^2 + xy + y^2) = C$; (*b*) $y^2 = 6x/(2x^3 + 3x^2 + C)$;
(*c*) $x + y = Ce^{-4x\,(2x+y)}$.

27. (*a*) $y = x - 3 - \dfrac{4(x - 2)}{\log_e C(x - 2)}$; (*b*) $y = x + 1 + C\sqrt{(1 + x^2)}$;
(*c*) $x^2y^2(y^2 - x^2) = C$.

28. (*a*) $y = 1/(2 - x^2 + Ce^{-x^2/2})$;
(*b*) $\log_e(x^2 + y^2 - 4x + 2y + 5) + 3\tan^{-1}\dfrac{y + 1}{x - 2} = C$;
(*c*) $y^2 = 3\cos^2 x/(C - 6\tan x - 2\tan^3 x)$.

29. (*a*) $\dfrac{1}{y^{1/2}} = \dfrac{1}{x}\{1 + Ce^{-x^{n+1}/2(n+1)}\}$.
(*b*) $2(x - 1)^2 + 2(x - 1)(y + 2) + 3(y + 2)^2 = C$.
(*c*) $10y - 5x + 7\log_e(10x + 5y - 1) = C$.

30. (*a*) $1/y = \sin x\{C - \cos x - \log_e(\tan \frac{1}{2}x)\}$;
(*b*) $\log_e\{(x - y)^4(x^2 + xy + y^2)\} = 2\sqrt{3}\tan^{-1}\dfrac{x + 2y}{\sqrt{3}.x} + C$;
(*c*) $\log_e\{(y - 1)^2 + 2(x - 1)^2\} = \sqrt{2}\tan^{-1}\dfrac{y - 1}{(x - 1)\sqrt{2}} + C$.

31. (*a*) $y = \dfrac{1}{2\sqrt{(1+x)}}\{Ce^{x} - (x^2 + 2x + 2)\}$;

(*b*) $y^2 = 5x^2(x^2 - 1)/2(x^5 - C)$; (*c*) $y + \sqrt{(x^2 + y^2)} = Cx^2$.

32. (*a*) $\log_e(y - x - 1) + \dfrac{2(x-1)}{y-x-1} = C$; (*b*) $y = \dfrac{C}{x}e^{1/x}$;

(*c*) $(y-2)^3 + 3(y-2)(x-1)^2 + 2(x-1)^3 = C$.

33. $y^2 = \cos^2 x/(4 \sin x + C)$.

34. $y - x\,dy/dx = \sqrt{(x^2 + y^2)}$; $r = C \operatorname{cosec}^2 \tfrac{1}{2}\theta$.

36. $x + y\,dy/dx = \sqrt{(x^2 + y^2)}$, $x - \sqrt{(x^2 + y^2)} = C$;

$\dfrac{dy}{dx} = \dfrac{y}{x - \sqrt{(x^2 + y^2)}}$, $C^2y^2 = 1 - 2Cx$.

38. $x\,dy/dx - y = x$, $y = x \log_e Cx$; $y = x\{1 + \log_e x\}$.

39. $x^2 - y^2 = C$; $(x\sqrt{5} - x) + 2\sqrt{(x^2 + y^2)} + C\{x\sqrt{5} + x - 2\sqrt{(x^2 + y^2)}\}$

$\{x\sqrt{(x^2 + y^2)} - y^2\}^{\sqrt{5}} = 0$.

40. Curve $ky = \cosh(kx + C)$;

Orthogonal curve $ky\sqrt{(k^2y^2 - 1)} - \cosh^{-1} ky = \pm 2k(x + C)$.

41. $y^2 = x^2(2x^2 - 1)$.

CHAPTER XVI (p. 500)

1. (i) $y = 7 - 2x^2$; (ii) $y = e^{-3x}(A \cos x + B \sin x) + \dfrac{3 \sin x - 2 \cos x}{39}$.

2. (i) $y = Ae^{-x} + Be^{-2x} - \dfrac{e^{-x}}{10}(\cos 2x + 2 \sin 2x)$; (ii) $x = \tan^{-1} y$.

3. (i) $y = A \cos 2x + B \sin 2x - \frac{1}{24} \cos 4x + (x/8) \sin 2x$;

(ii) $y = Ae^{-3x} + Be^{-4x} - 10(x + 1)e^{-4x}$.

4. (i) $y = (A + Bx)e^{2x} + \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{8}$;

(ii) $y = Ae^{x} + Be^{-x} + \frac{1}{20}(\cos 3x - 5 \cos x)$.

5. (*a*) $y = \frac{9}{5}\{2 \cos x + \sin x\} e^{-x} + \frac{1}{5}(\sin x - 2 \cos x)$;

(*b*) $y = (2 - 3x + \frac{1}{2}x^2)e^{x}$.

6. (i) $y = Ae^{-3x/2} + Be^{-x/2} - \frac{1}{2}e^{-x} \sin \frac{1}{2}x$;

(ii) $y = Ae^{2x} + Be^{-x} + \frac{1}{20}(\cos 2x - 3 \sin 2x)$.

7. (i) $y = Ae^{2x} + e^{-x}\{B - \frac{1}{9}(3x + 1) + \frac{1}{10}(3 \cos x - \sin x)\}$;

(ii) $y = (A + B \log_e x)x^{1/3} + \frac{1}{4}x(\log_e x - 3)$.

8. (i) $y = e^{3x}(A \cos 3x + B \sin 3x - \frac{1}{6}x \cos 3x)$;

(ii) $y = \dfrac{x^2}{16}(A + 4 \log_e x) + Bx^{-2}$.

9. (i) $y = (A + Bx + \frac{1}{2}x^2 + \frac{1}{6}x^3)e^{3x}$;
(ii) $y = Ax^4 + Bx^{-1} - \frac{1}{6}x^2 - \frac{1}{8}\{4 \log_e x - 3\}$.

10. (i) $y = e^{2x}(A \cos x + B \sin x - \frac{1}{2}x \cos x)$;
(ii) $y = x^2\{A + B \log_e x + \frac{1}{6}(\log_e x)^3\}$.

11. (i) $y = e^{-x/2}\{A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x\} + e^x\{\frac{1}{3}(x - 1) + \frac{2 \sin x - 3 \cos x\}}{13}$;

(ii) $y = Ax + Bx^{-1/4} + \frac{x}{25}(5 \log_e x - 4) - \log_e x + 3$.

12. (i) $y = \frac{e^x}{8}(A - \cos 2x - \sin 2x) + Be^{3x}$; (ii) $y = (A + Bx)e^{2x} + \frac{1}{8}\cos 2x$.

14. $y = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} + \sin t - 2 \cos t$.

15. (i) $y = e^{-x}[A \cos 2x + B \sin 2x] + \frac{13}{125} + \frac{x}{25} + \frac{x^2}{5}$;

(ii) $y = \frac{e^{-x}}{17}(4 \cos 2x + \sin 2x) + \frac{1}{17}(\sin 2x - 4 \cos 2x)$; (iii) $y = 1 + x^3$.

16. (i) $y = Ae^{-3x} + Be^{-2x} - \frac{e^{-2x}}{10}(\cos 2x + 2 \sin 2x)$;

(ii) $y = e^{-ax}\left(A + Bx + \frac{x^4}{12}\right)$.

17. (i) $x = t^2$; (ii) $y = (1 + \alpha x + x^4)e^{-\alpha x}$.

18. $A(12 + 6x + x^2)$.

19. $(-1)^r(r + 1)a_0$; $a_0/(1 + x)^2$.

20. $Ax(3 + 4x^2)$; $\frac{a_{2r+2}}{a_{2r}} = \frac{9 - 4r^2}{2(r + 1)(2r + 1)}$.

21. $\sum_{n=0}^{\infty} a_n x^n$, where $\frac{a_{n+1}}{a_n} = \frac{-2}{(n + 1)(2n + 3)}$.

22. $d^2y/dz^2 - 2y = 2(1 - z^2)$; $y = Ae^{\sqrt{2}\sin x} + Be^{-\sqrt{2}\sin x} + \sin^2 x$.

23. $y = A\left(\frac{1 - x^2}{1 + x^2}\right) + 2B\frac{x}{1 + x^2}$.

24. $x = e^{-t}(A \cos t + B \sin t) + \frac{e^t}{25}(15t - 2)$,

$y = e^{-t}(B \cos t - A \sin t) + \frac{e^t}{25}(5t + 11)$.

25. $x = e^t + e^{-t}$, $y = e^{-t} - e^t + \sin t$.

26. $x = \frac{1}{2}(e^{-3t} + e^{-t})$, $y = \frac{1}{2}(e^{-3t} - e^{-t})$.

28. $u = A \sin(\omega t + \varepsilon)$, $v = E/H - A \cos(\omega t + \varepsilon)$, where $\omega = eH/m$,

$x = B - \frac{A}{\omega}\cos(\omega t + \varepsilon)$, $y = C + \frac{E}{H}t - \frac{A}{\omega}\sin(\omega t + \varepsilon)$.

29. $x = \frac{6}{7}(e^{-2t} - e^{-3t/5}) + \sin t,\ y = 2\cos t - \frac{2}{7}(6e^{-2t} + e^{-3t/5})$.

30. $x = \frac{1}{27}\{3t + 1 - e^{-3t}(1 + 6t)\},\ y = \frac{1}{27}\{4 - 6t - e^{-3t}(4 + 6t)\}$.

31. $x = \frac{3}{5} + \frac{1}{6}e^{t} - \frac{3}{4}e^{-t} - \frac{1}{60}e^{-5t},\ y = \frac{2}{5} + \frac{1}{3}e^{t} - \frac{3}{4}e^{-t} + \frac{1}{60}e^{-5t}$.

32. $z = f(x + at) + \phi(x - at)$; $\quad z = \dfrac{a+1}{2a}e^{x+at} + \dfrac{a-1}{2a}e^{x-at} - 1$.

33. $\dfrac{\partial z}{\partial x}(2y^2 + xz) + \dfrac{\partial z}{\partial y}(2x^2 + yz) + (z^2 - 4xy) = 0$.

$$\left(\frac{\partial u}{\partial x}\right)_{y\ \text{const}} \times \left(\frac{\partial v}{\partial y}\right)_{x\ \text{const}} = \left(\frac{\partial u}{\partial y}\right)_{x\ \text{const}} \left(\frac{\partial v}{\partial x}\right)_{y\ \text{const.}}$$

34. $z = \cos y(1 + \cos x)$; $\quad z = f(x + y) + \phi(x - y) - \frac{1}{4}\sin(x - y)\cos(x + y)$.

35. $Y = Ce^{-a^2p^2y}$. **36.** $\dfrac{\partial^2 u}{\partial \theta^2} = \dfrac{\partial u}{\partial z}$.

37. $\phi(t) = A\cos ct + B\sin ct$.

38. $Y = Ae^{mv} + Be^{-mv}$; $\quad V = \dfrac{1}{1 - e^4}\{e^{1-v} - e^{3+v}\}\sin x$.

39. (i) $V = A + B\log_e r$; (ii) $V = r^n(A\cos n\theta + B\sin n\theta)$.

40. (i) $u = A + B\log_e r$; (ii) $\alpha = \pm 1$.

41. $\dfrac{\partial V}{\partial t} = \dfrac{-x}{4t^{3/2}}f'(z)$; $\quad \dfrac{\partial^2 V}{\partial x^2} = \dfrac{1}{4t}f''(z)$.

42. $i = E\left\{\left(\dfrac{1}{C} - Lp^2\right)\cos pt + pR\sin pt\right\} \Big/ \left\{\left(\dfrac{1}{C} - Lp^2\right)^2 + p^2R^2\right\}$.

43. Particular integral $= \dfrac{kt}{2p}e^{-ht}\sin pt$; amplitude $= \dfrac{kt}{2p}e^{-ht}$.

44. (i) $R^2 < 4L/C$.

(ii) $i = e^{-Rt/2L}\left\{A\cos\dfrac{N}{2L}t + B\sin\dfrac{N}{2L}t\right\} + \dfrac{E\left\{\left(\dfrac{1}{C} - Lp^2\right)\cos pt + pR\sin pt\right.}{\left(\dfrac{1}{C} - Lp^2\right)^2 + p^2R^2}$.

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